The Capital Asset Pricing Model

Lecture Outline

- Optimal Portfolio Choice
- The CAPM
- The Capital Market Line and Security Market Line
- Systematic Risk and Beta

The Expected Return of a Portfolio

- Portfolio Weights
  - The fraction of the total investment in the portfolio held in each individual investment in the portfolio
    - The portfolio weights must add up to 1.00 or 100%.

$$x_i = \frac{\text{Value of investment } i}{\text{Total value of portfolio}}$$

- Then the return on the portfolio, R_p , is the weighted average of the returns on the investments in the portfolio, where the weights correspond to portfolio weights.

$$R_P = x_1 R_1 + x_2 R_2 + \cdots + x_n R_n = \sum_i x_i R_i$$

- The expected return of a portfolio is the weighted average of the expected returns of the investments within it.

$$E[R_P] = E\left[\sum_i x_i R_i\right] = \sum_i E[x_i R_i] = \sum_i x_i E[R_i]$$

- **Problem**
  - Assume your portfolio consists of \$25,000 of Intel stock and \$35,000 of ATP Oil and Gas.
  - Your expected return is 18% for Intel and 25% for ATP Oil and Gas.
  - **What is the expected return for your portfolio?**

- **Solution**
  - Total Portfolio = \$25,000 + 35,000= \$60,000
  - Portfolio Weights
    - Intel: \$25,000 $\div$ \$60,000 = .4167
    - ATP: \$35,000 $\div$ \$60,000 = .5833
  - Expected Return
    - E[R] = (.4167)(.18) + (.5833)(.25)
    - E[R] = 0.075006 + 0.145825 = 0.220885 = 22.1%

The Volatility of a Two-Stock Portfolio

Combining Risks

Table: Returns for Three Stocks, and Portfolios of Pairs of Stocks

	Stock Returns			Portfolio Returns	
Year	North Air	West Air	Tex Oil	$1/2R_N + 1/2R_W$	$1/2R_W + 1/2R_T$
2003	21%	9%	–2%	15.0%	3.5%
2004	30%	21%	–5%	25.5%	8.0%
2005	7%	7%	9%	7.0%	8.0%
2006	–5%	–2%	21%	–3.5%	9.5%
2007	–2%	–5%	30%	–3.5%	12.5%
2008	9%	30%	7%	19.5%	18.5%
Average Return	10.0%	10.0%	10.0%	10.0%	10.0%
Volatility	13.4%	13.4%	13.4%	12.1%	5.1%

- Combining Risks
  - By combining stocks into a portfolio, we reduce risk through diversification.
  - The amount of risk that is eliminated in a portfolio depends on the degree to which the stocks face common risks and their prices move together.
  - To find the risk of a portfolio, one must know the degree to which the stocks' returns move together.

Determining Covariance and Correlation

- Covariance
  - The expected product of the deviations of two returns from their means
  - Covariance between Returns R_i and R_j

$$Cov(R_i, R_j) = E[(R_i - E[R_i])(R_j - E[R_j])]$$

  - Estimate of the Covariance from Historical Data

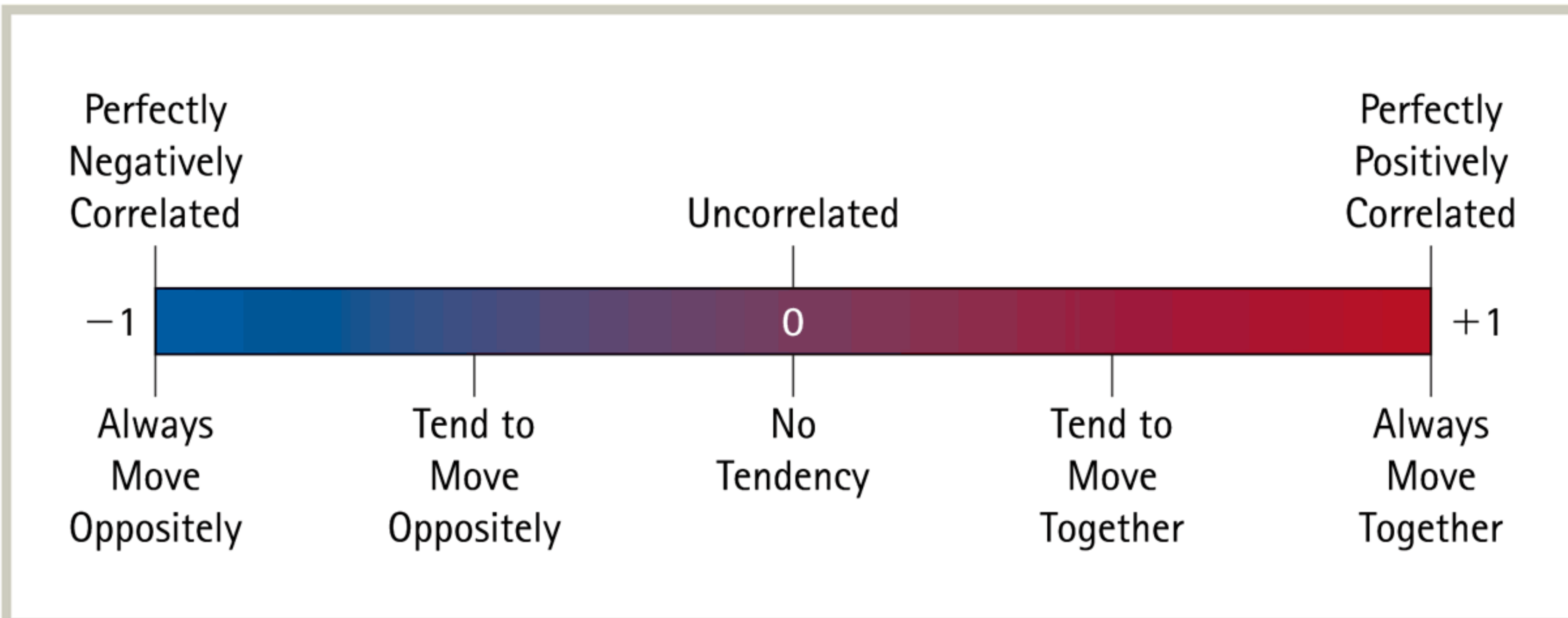
$$Cov(R_i, R_j) = \frac{1}{T-1} \sum_t (R_{i,t} - \overline{R}_i)(R_{j,t} - \overline{R}_j)$$

    - If the covariance is positive, the two returns tend to move together.
    - If the covariance is negative, the two returns tend to move in opposite directions.

- Correlation
  - A measure of the common risk shared by stocks that does not depend on their volatility

$$Corr(R_i, R_j) = \frac{Cov(R_i, R_j)}{SD(R_i)\, SD(R_j)}$$

    - The correlation between two stocks will always be between –1 and +1.


Perfectly
Negatively
Correlated
Uncorrelated
Perfectly
Positively
Correlated
−1
0
+1
Always
Move
Oppositely
Tend to
Move
Oppositely
No
Tendency
Tend to
Move
Together
Always
Move
Together

- **Computing the Covariance and Correlation Between Pairs of Stocks**

	Deviation from Mean			North Air and West Air	West Air and Tex Oil
Year	$(R_N - \bar{R}_N)$	$(R_W - \bar{R}_W)$	$(R_T - \bar{R}_T)$	$(R_N - \bar{R}_N)(R_W - \bar{R}_W)$	$(R_W - \bar{R}_W)(R_T - \bar{R}_T)$
2003	11%	−1%	−12%	−0.0011	0.0012
2004	20%	11%	−15%	0.0220	−0.0165
2005	−3%	−3%	−1%	0.0009	0.0003
2006	−15%	−12%	11%	0.0180	−0.0132
2007	−12%	−15%	20%	0.0180	−0.0300
2008	−1%	20%	−3%	−0.0020	−0.0060
	$\text{Sum} = \sum_t (R_{i,t} - \bar{R}_i)(R_{j,t} - \bar{R}_j) =$			0.0558	−0.0642
Covariance:	$Cov(R_i, R_j) = \frac{1}{T-1}\text{Sum} =$			0.0112	−0.0128
Correlation:	$Corr(R_i, R_j) = \frac{Cov(R_i, R_j)}{SD(R_i)\,SD(R_j)} =$			0.624	−0.713

Computing a Portfolio's Variance and Volatility

- For a two security portfolio:

$$
\begin{aligned}
Var(R_P) &= Cov(R_P, R_P) \\
&= Cov(x_1R_1 + x_2R_2, x_1R_1 + x_2R_2) \\
&= x_1x_1Cov(R_1, R_1) + x_1x_2Cov(R_1, R_2) + x_2x_1Cov(R_2, R_1) + x_2x_2Cov(R_2, R_2)
\end{aligned}
$$

  - The Variance of a Two-Stock Portfolio

$$Var(R_P) = x_1^2Var(R_1) + x_2^2Var(R_2) + 2x_1x_2Cov(R_1, R_2)$$

- **Problem**
  - Assume your portfolio consists of $25,000 of Intel stock and $35,000 of ATP Oil and Gas.
  - Your expected return is 18% for Intel and 25% for ATP Oil and Gas. Assume the annual standard deviation of returns is 43% for Intel and 68% for ATP Oil and Gas.
  - If the correlation between Intel and ATP is .49, what is the standard deviation of your portfolio?

- Solution

$$SD(R_P) = \sqrt{x_1^2 Var(R_1) + x_2^2 Var(R_2) + 2x_1 x_2 Cov(R_1, R_2)}$$

$$SD(R_P) = \sqrt{(.4167)^2(.43)^2 + (.5833)^2(.68)^2 + 2(.4167)(.5833)(.49)(.43)(.68)}$$

$$SD(R_P) = \sqrt{(.1736)(.1849) + (.3402)(.4624) + 2(.4167)(.5833)(.49)(.43)(.68)}$$

$$SD(R_P) = \sqrt{.0321 + .1573 + .0696} = \sqrt{0.259} = .5089 = 50.89\%$$

The Volatility of a Large Portfolio

- *The variance of a portfolio is equal to the weighted average covariance of each stock with the portfolio:*

$$Var(R_P) = Cov(R_P, R_P) = Cov\left(\sum_i x_i R_i, R_P\right) = \sum_i x_i Cov(R_i, R_P)$$

  - which reduces to:

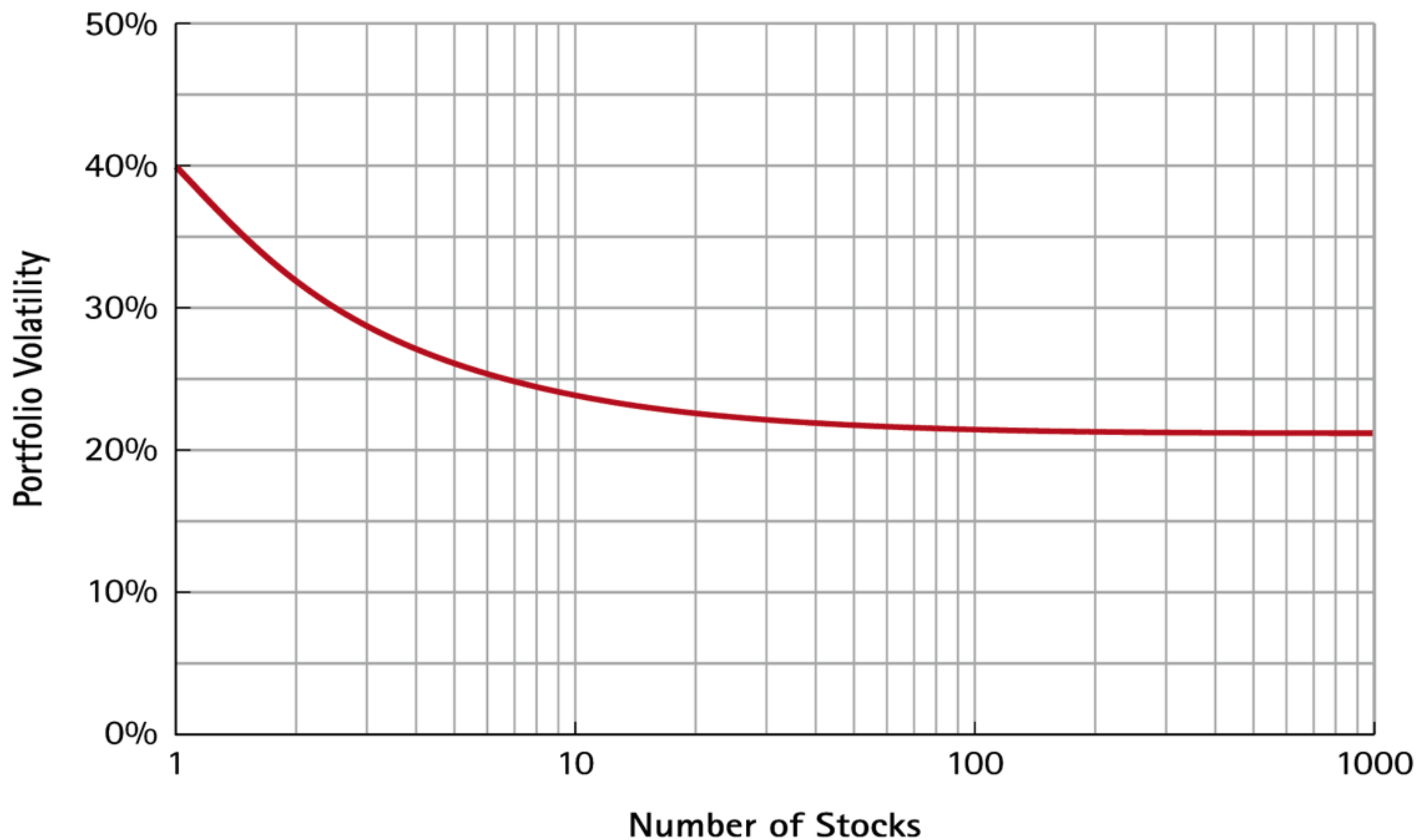
$$Var(R_P) = \sum_i x_i Cov(R_i, R_P) = \sum_i x_i Cov(R_i, \Sigma_j x_j R_j)$$
$$= \sum_i \sum_j x_i x_j Cov(R_i, R_j)$$

Diversification with an Equally Weighted Portfolio of Many Stocks

- Equally Weighted Portfolio
  - A portfolio in which the same amount is invested in each stock
- Variance of an Equally Weighted Portfolio
  of *n* Stocks

$$Var(R_P) = \frac{1}{n}(\text{Average Variance of the Individual Stocks})$$
$$+ \left(1 - \frac{1}{n}\right)(\text{Average Covariance between the Stocks})$$

Volatility of an Equally Weighted Portfolio Versus the Number of Stocks

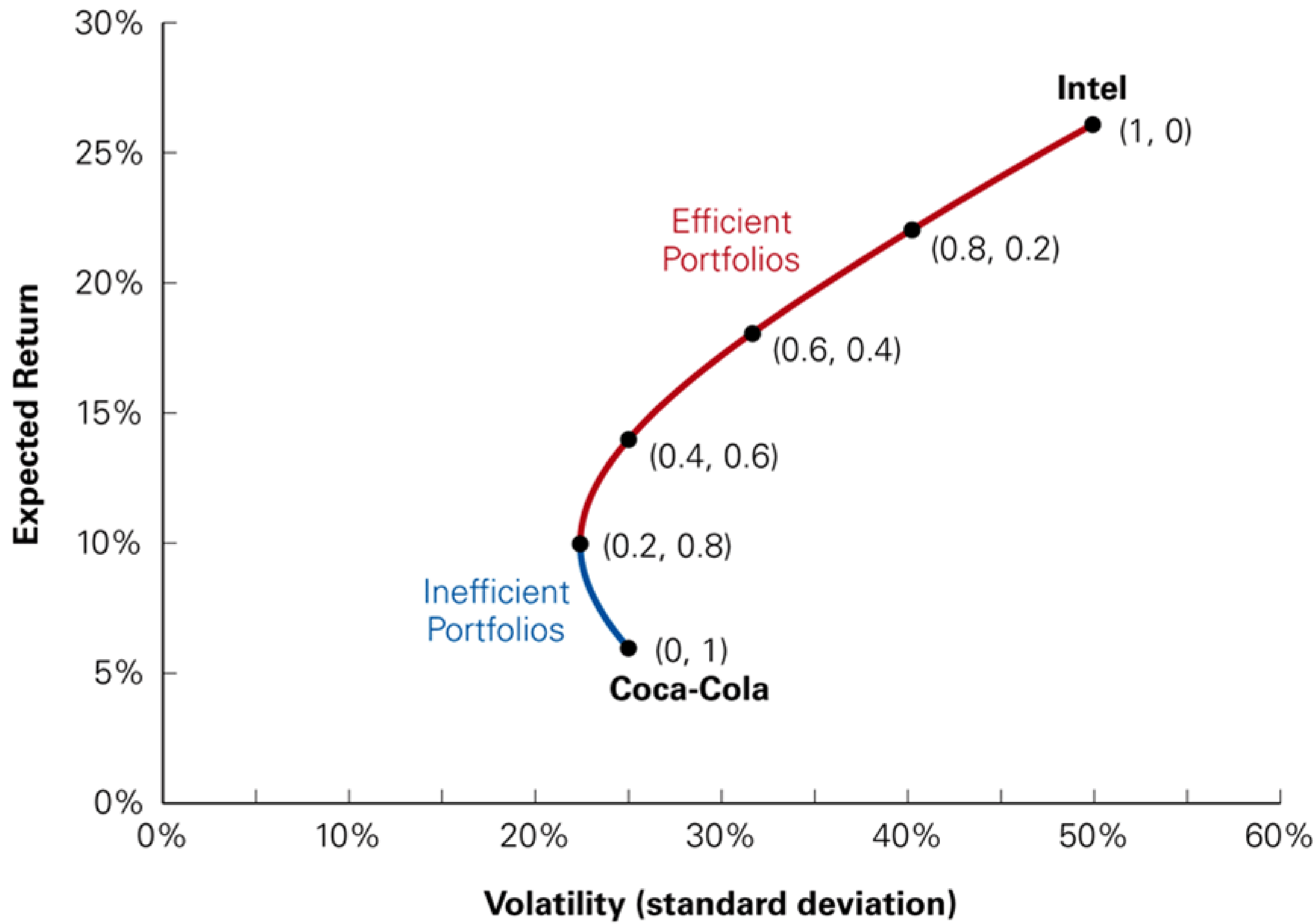


The Optimal Portfolio Choice

- By varying the portfolio weights, our portfolio achieves different combinations of risk (standard deviation) and expected return
- The trade-off between risk and expected return is represented by the opportunity/feasible set.
- Consider Intel and Coca Cola:

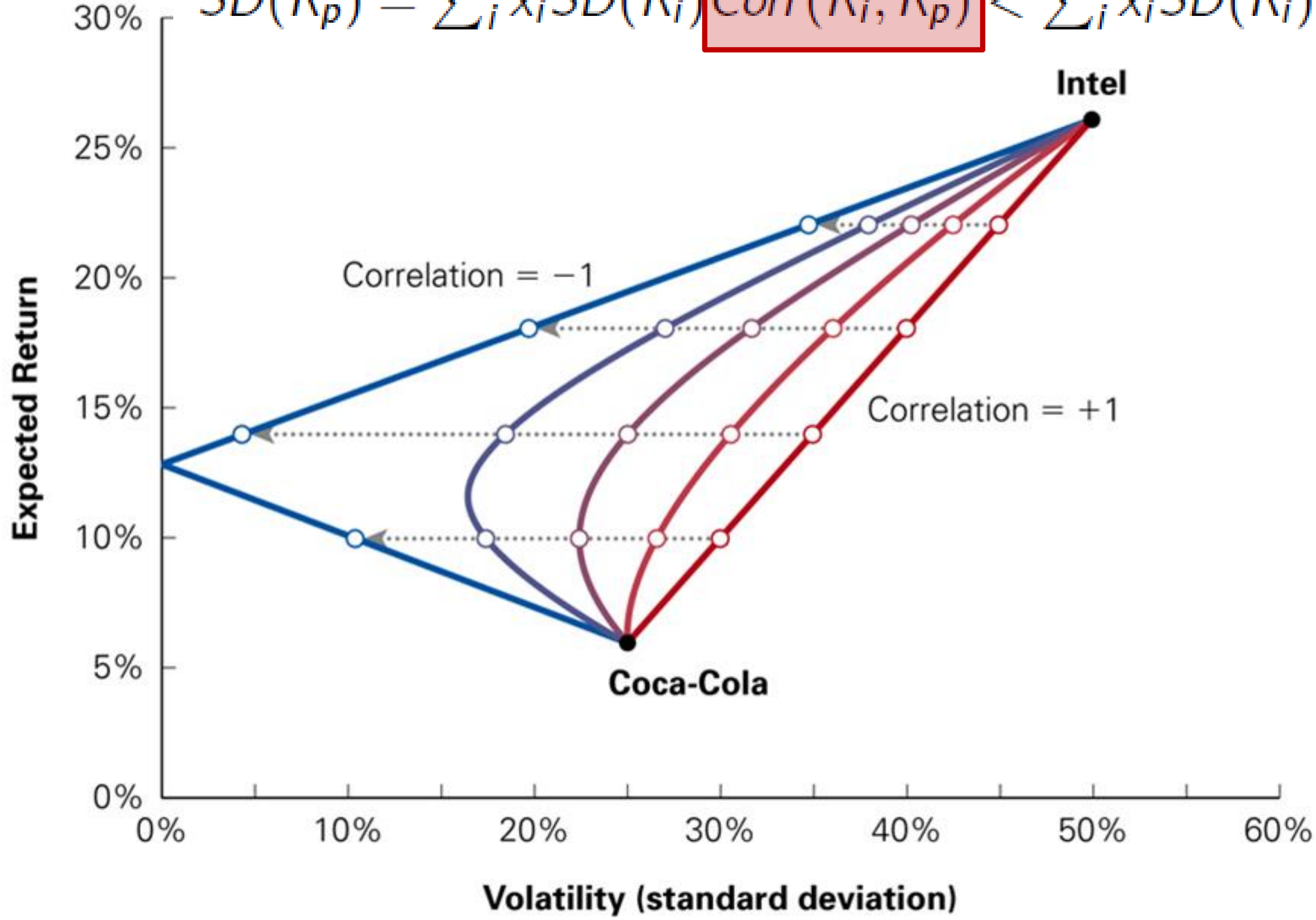
Portfolio Weights		Expected Return (%)	Volatility (%)
x_I	x_C	$E[R_P]$	$SD[R_P]$
1.00	0.00	26.0	50.0
0.80	0.20	22.0	40.3
0.60	0.40	18.0	31.6
0.40	0.60	14.0	25.0
0.20	0.80	10.0	22.3
0.00	1.00	6.0	25.0

Feasible Set



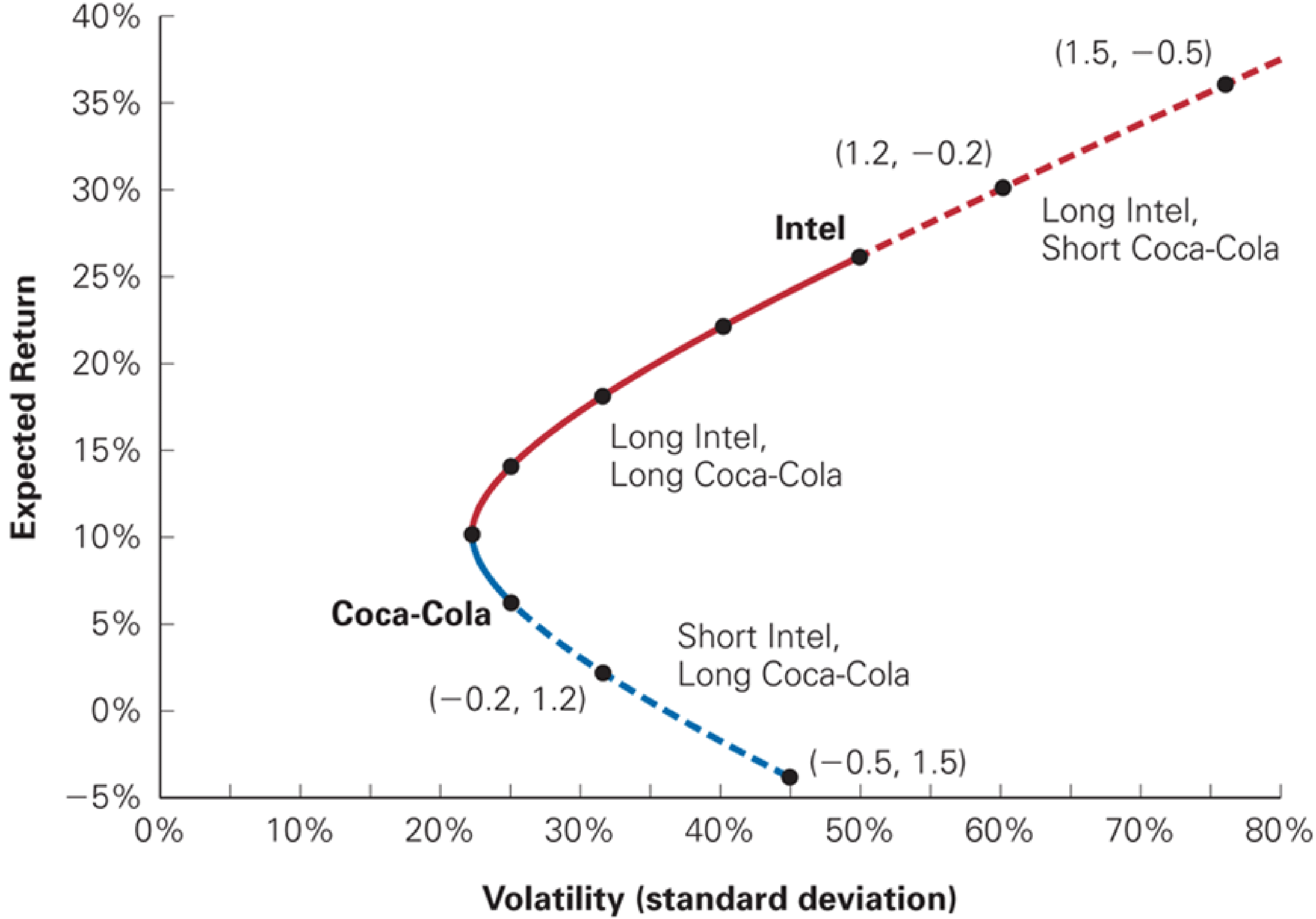
The effect of Correlation

$$SD(R_p) = \sum_i x_i SD(R_i) \boxed{Corr(R_i, R_p)} < \sum_i x_i SD(R_i)$$



- Correlation: No effect on portfolio expected return, but on portfolio volatility

Short Sales



40%
35%
30%
25%
20%
15%
10%
5%
0%
−5%
Expected Return
(1.5, −0.5)
(1.2, −0.2)
Intel
Long Intel,
Short Coca-Cola
Long Intel,
Long Coca-Cola
Coca-Cola
Short Intel,
Long Coca-Cola
(−0.2, 1.2)
(−0.5, 1.5)
0%
10%
20%
30%
40%
50%
60%
70%
80%
Volatility (standard deviation)

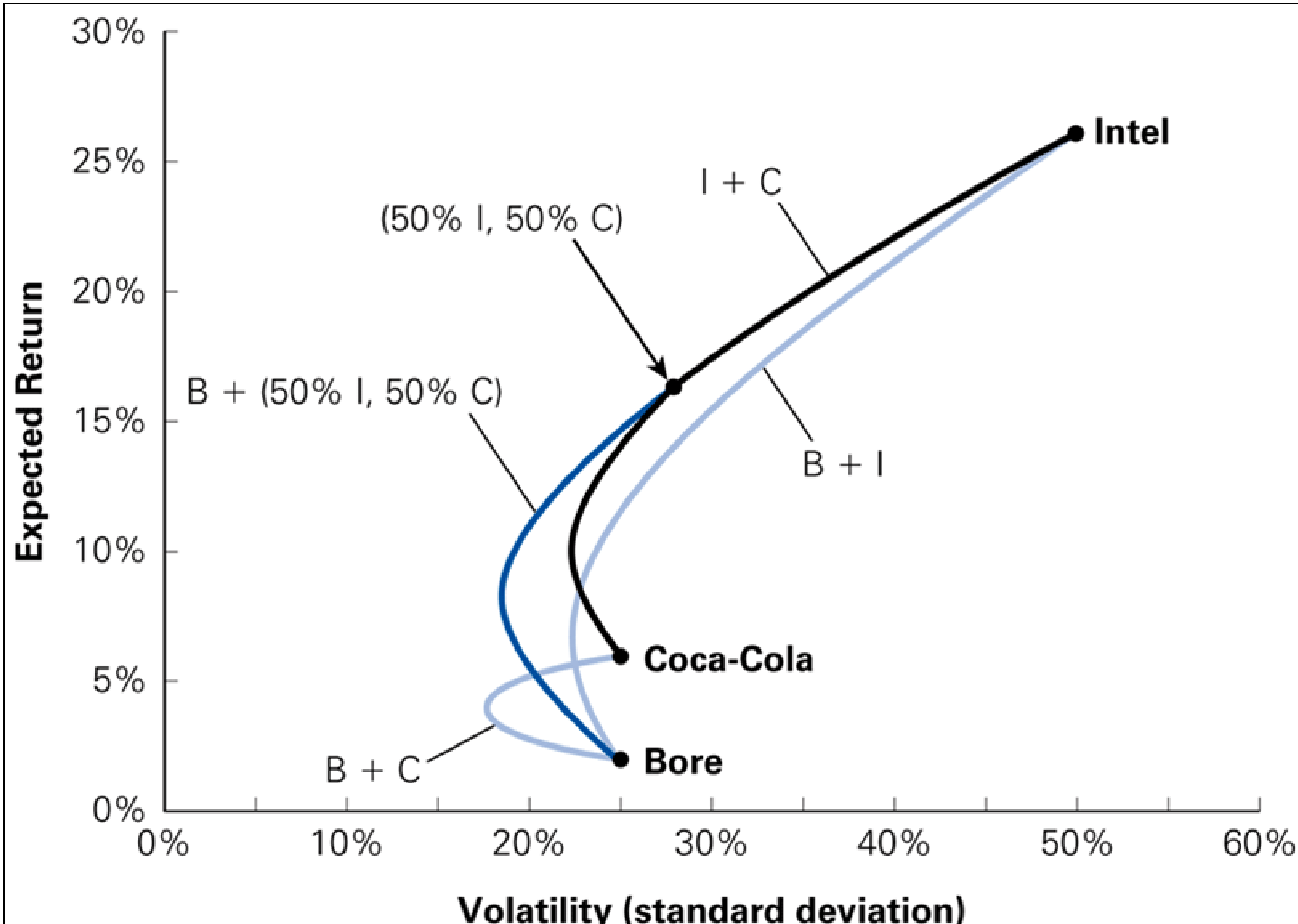
Efficient Portfolio with Many Stocks

- Consider adding Bore Industries into the two-stock portfolio:

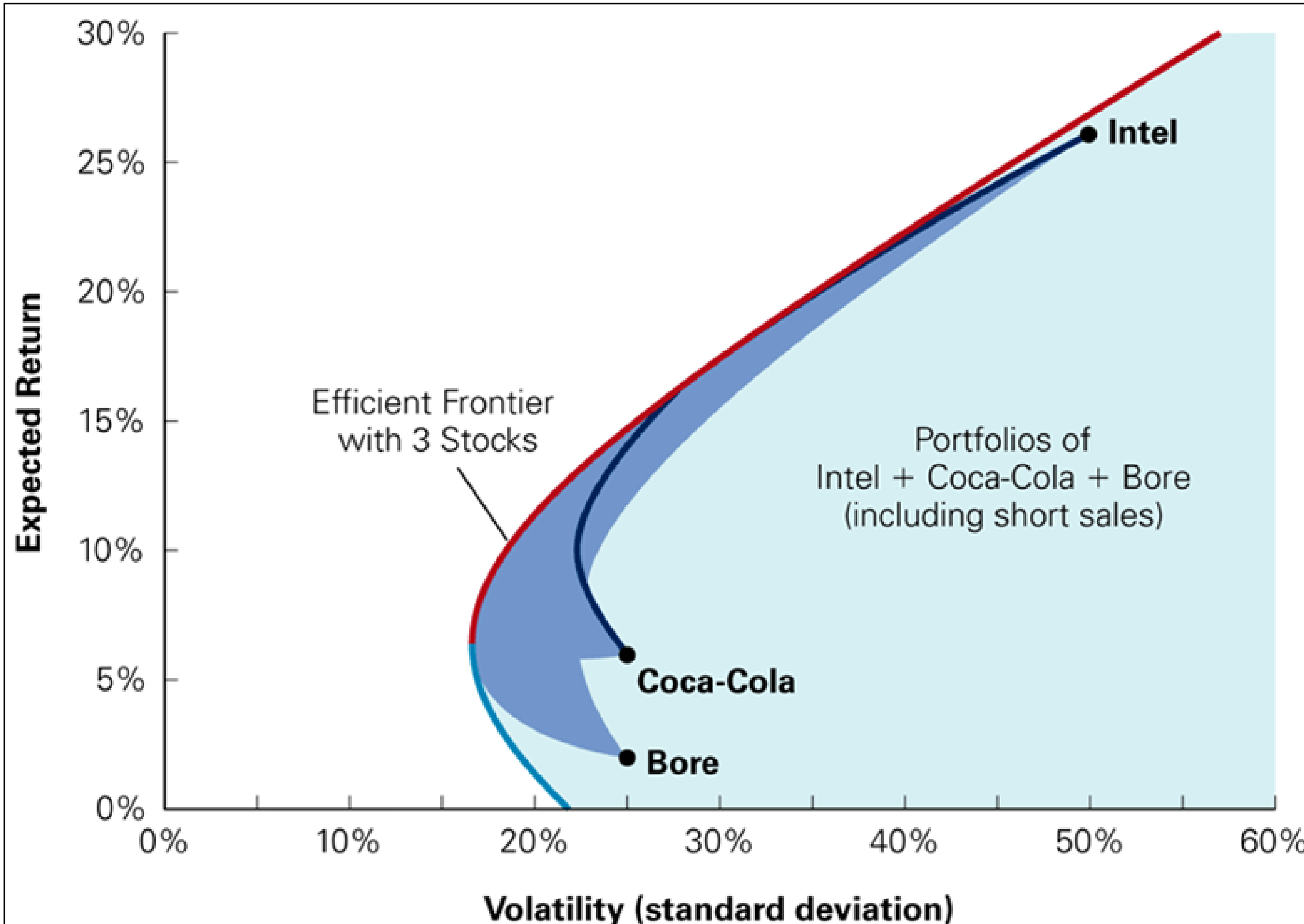
			Correlation with		
Stock	**Expected Return**	**Volatility**	**Intel**	**Coca-Cola**	**Bore Ind.**
Intel	26%	50%	1.0	0.0	0.0
Coca-Cola	6%	25%	0.0	1.0	0.0
Bore Industries	2%	25%	0.0	0.0	1.0

- Is it beneficial to add Bore to the portfolio?

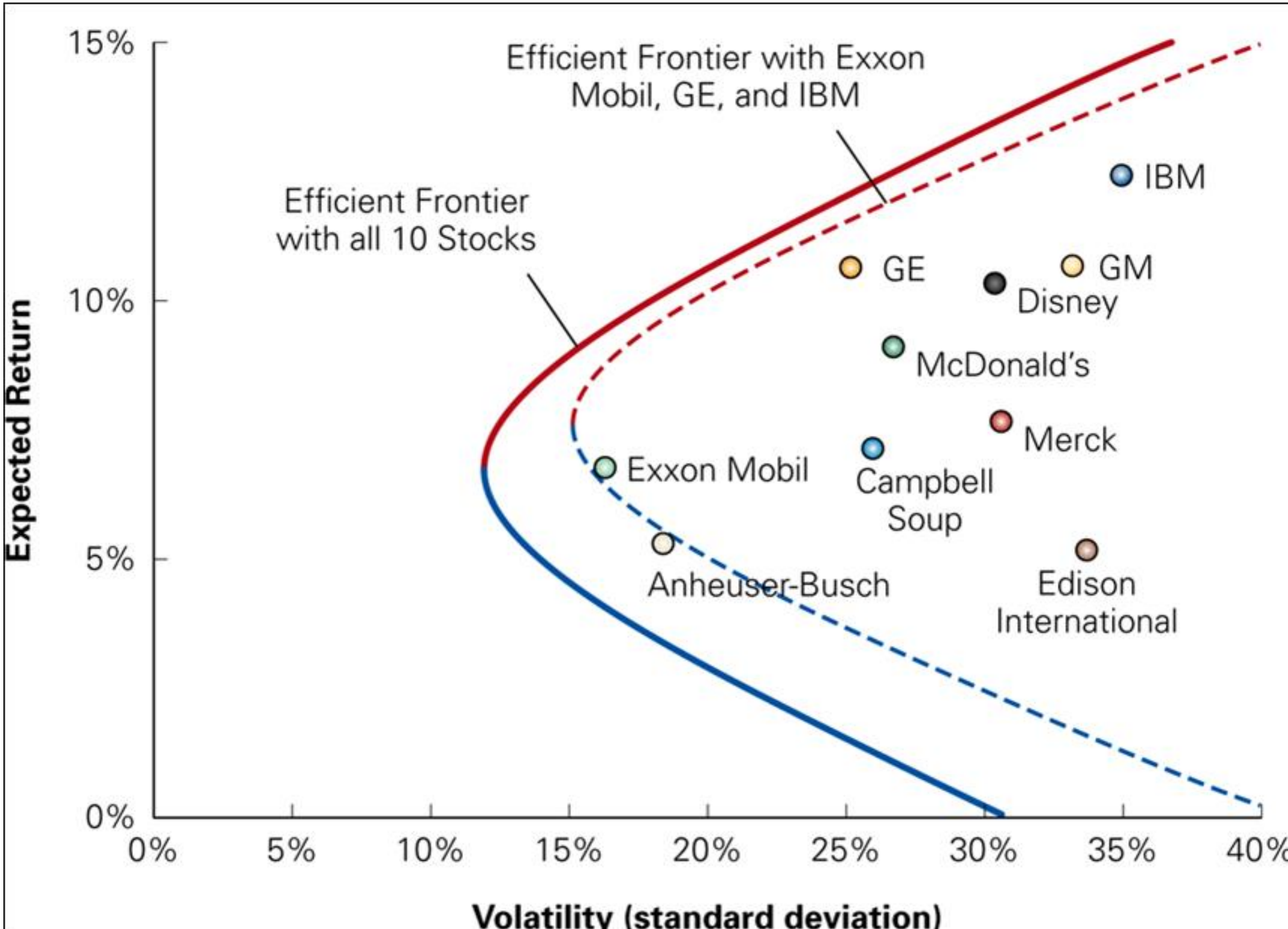
Expected Return and Volatility



Efficient Frontier (3-stock)



Efficient Frontier (10 vs 3-stock)



Risk-Free Saving and Borrowing

- Consider: x% in a risky portfolio, A; (1-x)% in risk-free Treasury bills
- The expected return of the portfolio:

$$E(R_p) = x \cdot E(R_A) + (1-x) \cdot r_f = r_f + x \cdot \big(E(R_A) - r_f\big)$$

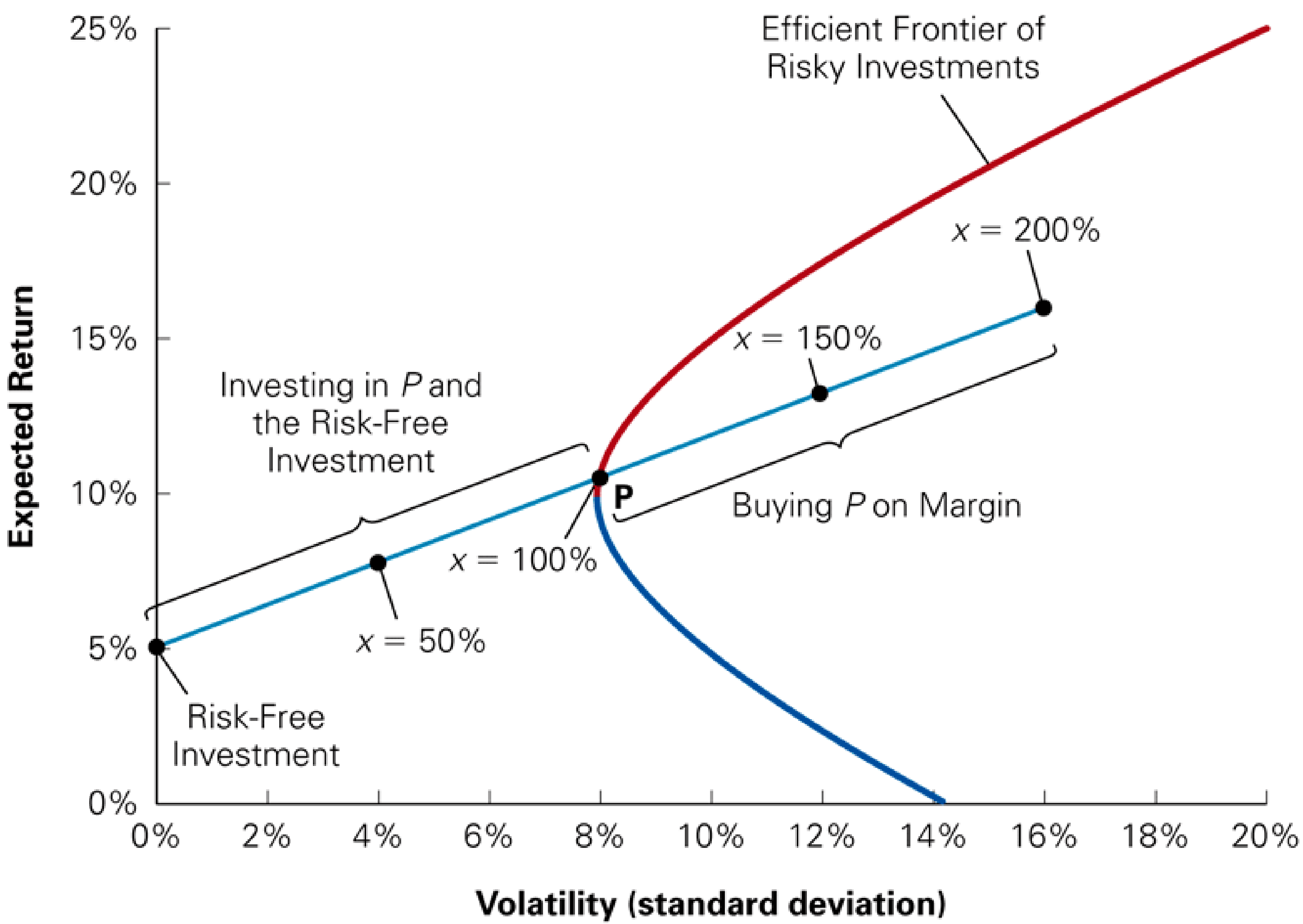
- The variance of the portfolio:

$$\sigma_p = \sqrt{x^2\sigma_A^2 + (1-x)^2\sigma_f^2 + 2 \cdot x(1-x)\sigma_{A,f}} = \sqrt{x^2\sigma_A^2} = x\sigma_A$$

- Linear relationship between the expected return and the SD of the portfolio:

$$E(r_p) = r_f + \left(\frac{E(R_A) - r_f}{\sigma_A}\right)\sigma_p$$

Efficient Frontier of Risky Investments

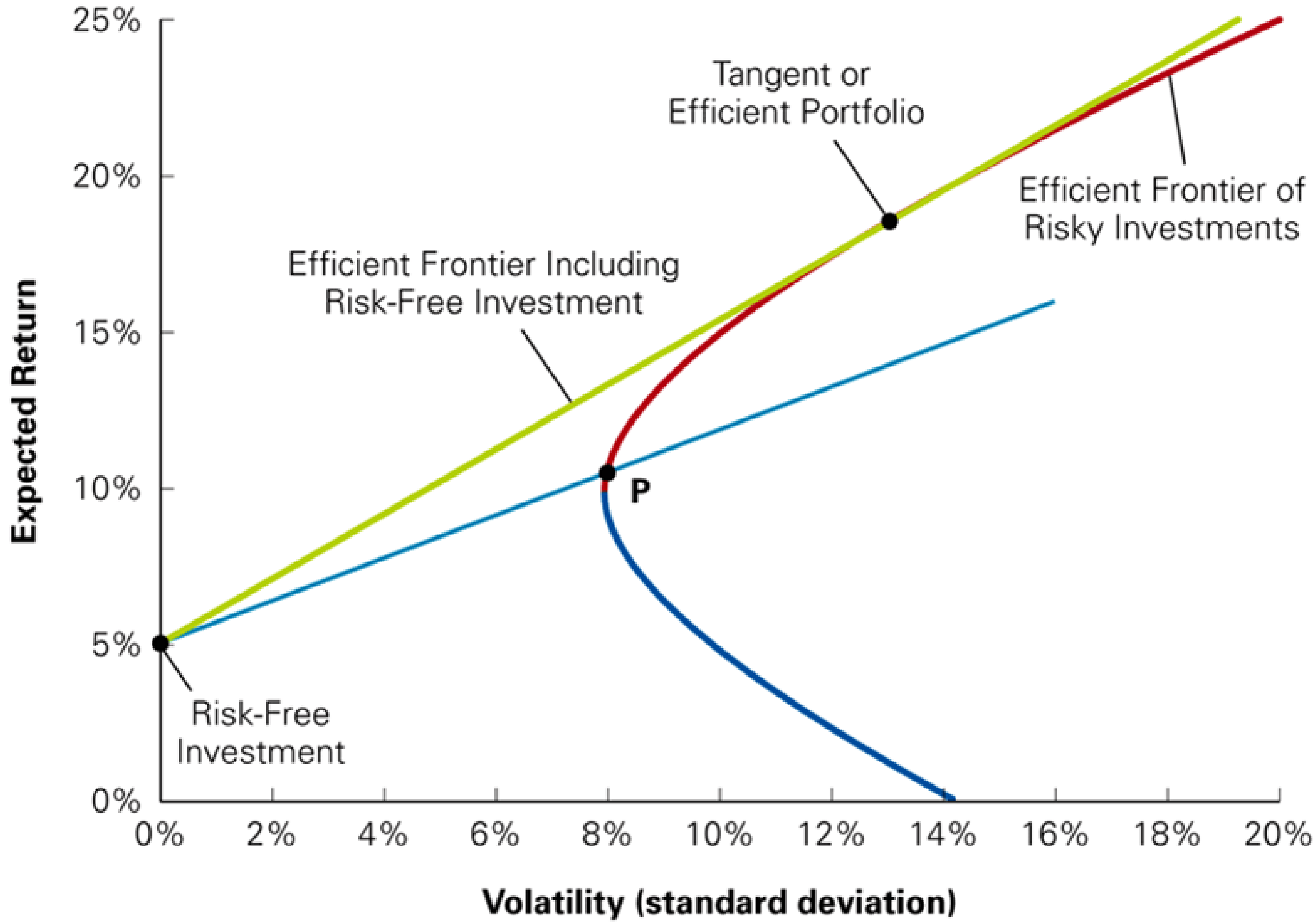


Tangent Portfolio

- The portfolio that earns the highest possible expected return for any level of volatility (find the steepest line when combined with the risk-free investments)
- Highest Sharpe ratio (the ratio of reward-to-volatility provided by a portfolio)

$$\text{Sharpe Ratio} = \frac{\text{Excess Return}}{\text{Standard Deviation}} = \frac{E(R_A) - r_f}{\sigma_A}$$

Tangent Portfolio



Tangent Portfolio Implications

- Combinations of the risk-free assets and the tangent portfolio provide the best risk return trade-off (efficient porfolio)
  - Every investors should invest in the tangent portfolio, independent of their taste for risk
- An investor's preference will only determine the allocation between tangent portfolio v the risk-free investment
  - Aggressive: more in tangent portfolio
  - Conservative: less in tangent portfolio

Quick Quiz

- Your uncle asks for investment advice. Currently, he has $100,000 invested in portfolio P in the figure "Tangent Portfolio" which has an expected return of 10.5% and a volatility of 8%. Suppose the risk-free rate is 5%, and the tangent portfolio has an expected return of 18.5% and a volatility of 13%. To maximize his expected return without increasing his volatility, which portfolio would you recommend? If your uncle prefers to keep his expected return the same but minimize his risk, which portfolio would you recommend?

Capital Asset Pricing Model

- Assume there is a portfolio of risky securities, *p*. Can you raise Sharpe ratio by adding more of some investment *i* to the portfolio?
- Recall that the contribution of investment *i* to the volatility of the portfolio depends on the risk that *i* has in common with the portfolio, which is measured by *i*'s volatility x its correlation with *p*.

$$SD(R_p) = \sum_i \underbrace{x_i \cdot SD(R_i) \cdot Corr(R_i, R_p)}_{\text{Security } i\text{'s contribution to the volatility of the portfolio}}$$

Capital Asset Pricing Model

- If you were to buy more of investment *i* by borrowing, you would earn the expected return of *i* minus the risk-free return. Adding *i* to the portfolio *p* improves the portfolio's Sharpe ratio if:

$$\underbrace{E(R_i) - r_f}_{\substack{\text{Additional returns} \\ \text{from investment } i}} > \underbrace{SD(R_i) \times Corr(R_i, R_p)}_{\substack{\text{Incremental volatility} \\ \text{from investment } i}} \times \underbrace{\frac{E(R_p) - r_f}{SD(R_p)}}_{\substack{\text{Returns per unit of volatility} \\ \text{available from portfolio } p}}$$

Capital Asset Pricing Model

- Define Beta of investment *i* with respect to portfolio *p* as:

$$\beta_i^p = \frac{SD(R_i) \times Corr(R_i, R_p)}{SD(R_p)} = \frac{Cov(R_i, R_p)}{Var(R_p)}$$

- Therefore, the condition can be expressed as:

$$E(R_i) > r_f + \beta_i^p (E(R_p) - r_f)$$

Capital Asset Pricing Model

- Higher investment in i will increase the Sharpe ratio of a portfolio p if its expected return E[Ri] exceeds the required return *ri*, defined as:

$$r_i \equiv r_f + \beta_i^p \left(E(R_p) - r_f \right)$$

- *ri* is the expected return necessary to compensate for the risk investment *i* contributes to the portfolio
- A portfolio is efficient iff the expected return of every available security equals to its required return.

$$E(R_i) = r_i \equiv r_f + \beta_i^{eff} \left(E(R_{eff}) - r_f \right)$$

Quick Quiz:

- Assume you own a portfolio of 25 different stocks. You expect your portfolio will have a return of 12% and a standard deviation of 15%. A colleague suggests you add gold to your portfolio. Gold has an expected return of 8%, a standard deviation of 25%, and a correlation with your portfolio of -0.05. If the risk-free rate is 2%, will adding gold improve your portfolio's Sharpe ratio?

$$\beta_{gold} = \frac{SD(R_{gold}) \times Corr(R_{gold}, R_p)}{SD(R_p)} = \frac{25\% \times -0.05}{15\%} = -0.0833$$

$$r_i \equiv r_f + \beta_{gold}(E(R_p) - r_f) = 2\% - 0.0833 \times (12\% - 2\%) = 1.17\%$$ %,

adding gold to your portfolio will increase Sharpe

CAPM Assumptions

- Investors can buy and sell all securities at competitive market prices (without incurring taxes or transactions costs) and can borrow and lend at the risk-free interest rate.
- Investors hold only efficient portfolios of traded securities portfolios that yield the maximum expected return for a given level of volatility.
- Investors have homogeneous expectations regarding the volatilities, correlations, and expected returns of securities.

Market Equilibrium and The Efficiency of the Market Portfolio

- Given homogenous expectations, all investors will demand the same efficient (tangent) portfolio of risky securities
  - If every investors hold the same tangency portfolio of risky assets, the sum of all investors must equal the efficient (tangent) portfolio
- Supply of all risky securities is just the market portfolio

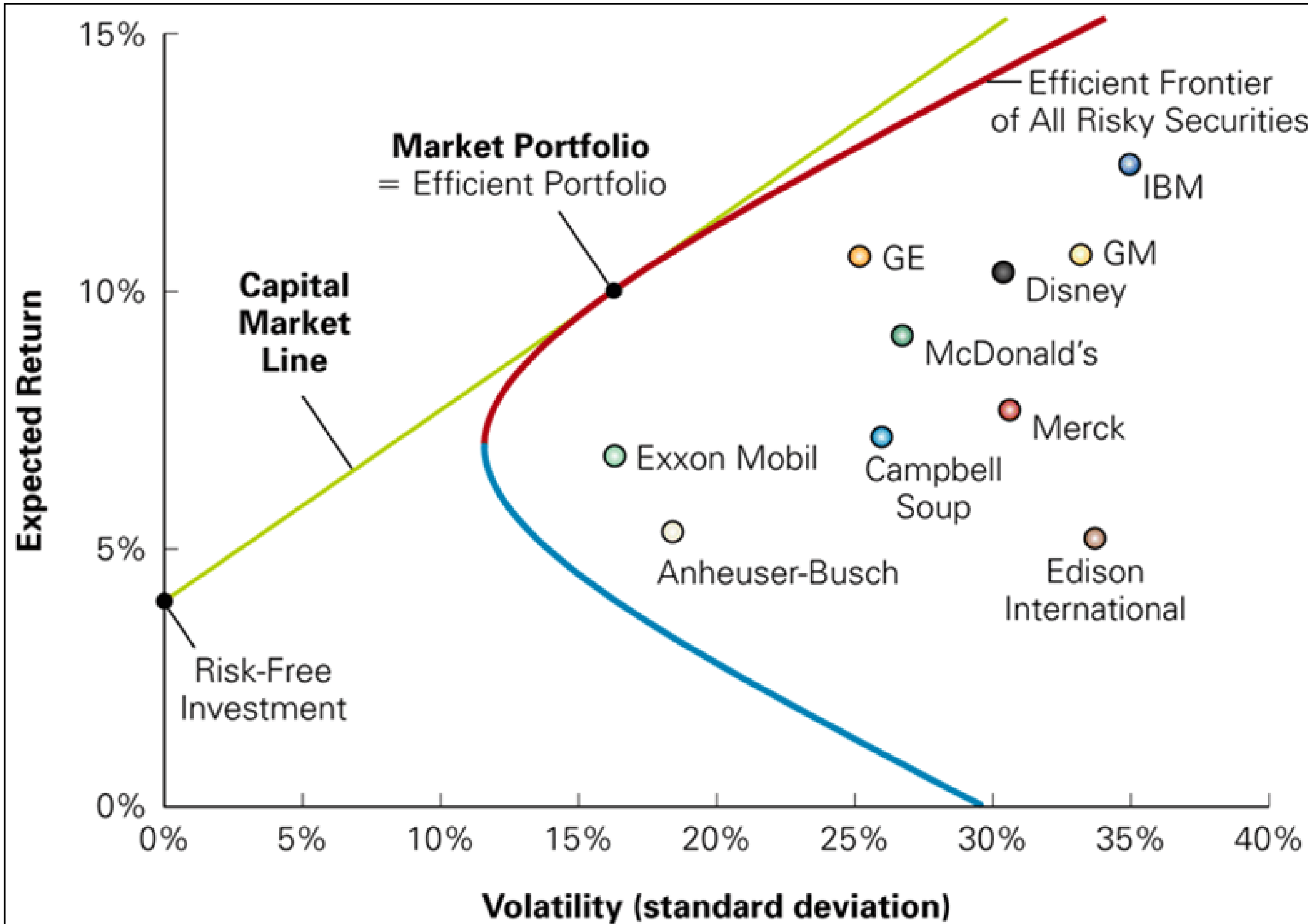
Market Equilibrium and The Efficiency of the Market Portfolio

- In equilibrium, demand must equal to supply. So, the efficient portfolio coincides with the market portfolio.
  - If a security in the market was not part of the efficient portfolio, then no investor would want to own it.
  - The security's price would fall, driving its expected return up until it became an attractive investment.

Capital Market Line

- When CAPM assumptions hold, an optimal (efficient) portfolio is a combination of the risk-free investment and the market portfolio
  - When the tangent line goes through the market portfolio, it's called the capital market line (CML)

Capital Market Line



The CAPM Equation

- Given an efficient market portfolio, the expected return of an individual security *i*:

$$E(R_i) = r_i \equiv r_f + \underbrace{\beta_i^m \left(E(R_m) - r_f\right)}_{\text{Risk premium for } i}$$

where $$\beta_i^m = \frac{SD(R_i) \times Corr(R_i, R_m)}{SD(R_m)} = \frac{Cov(R_i, R_m)}{Var(R_m)}$$

- The risk premium for an individual security *i*:

$$r_{\text{Premium}} = \beta_i^m \left(E(R_m) - r_f\right)$$

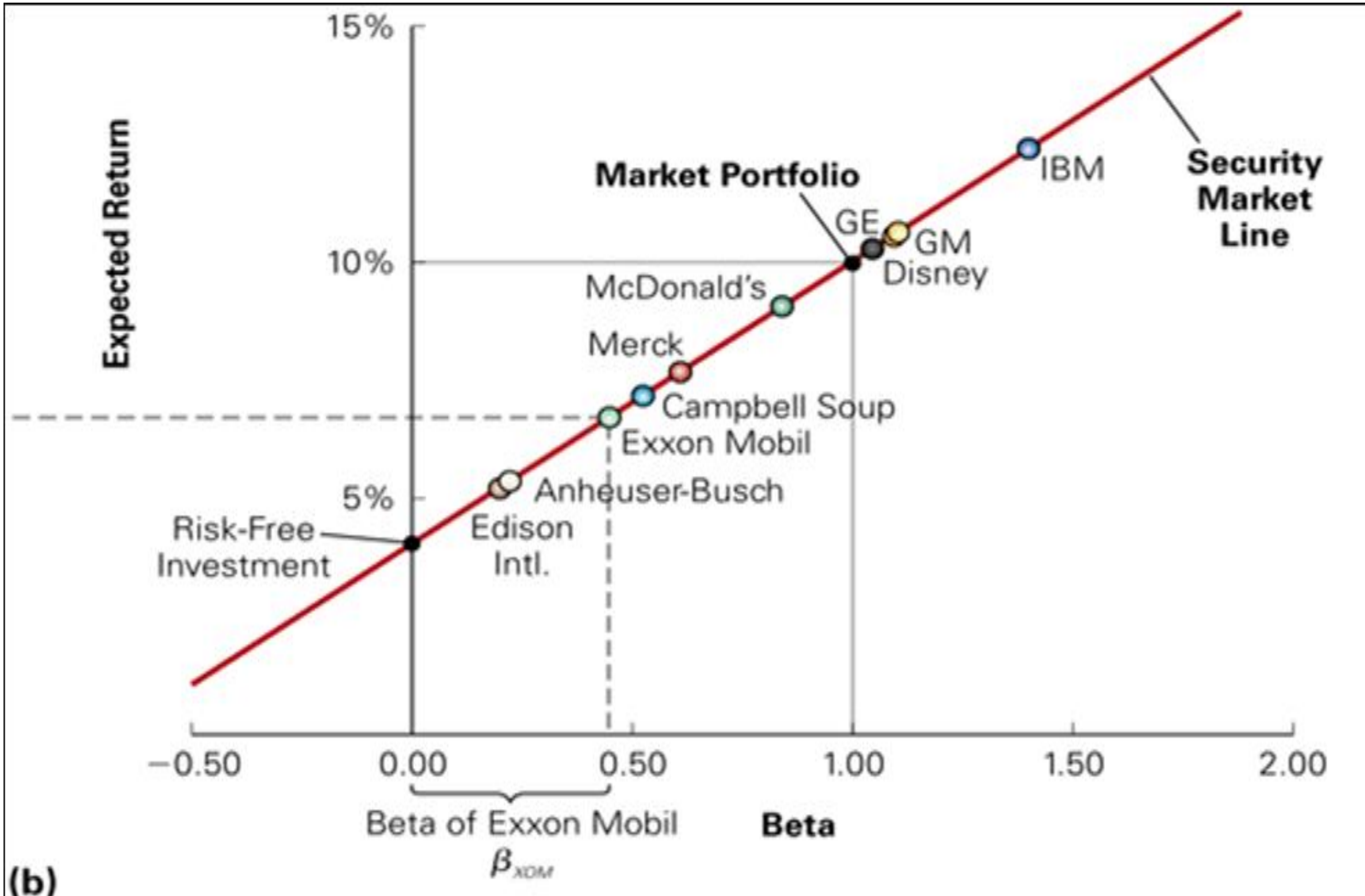
The Security Market Line

- The CAPM Formula:

$$E(R_i) = r_f + \boxed{\beta_i}\boxed{(E(R_m) - r_f)}$$

- Implies a linear relationship between an asset's expected return and its beta
- The plot of this relationship is called the Security Market Line (SML)
- Risk Premium
  - Market Price of Systematic Risk: $(E(R_m) - r_f)$
  - Quantity of Systematic Risk an asset adds into a well diversified portfolio: β_i

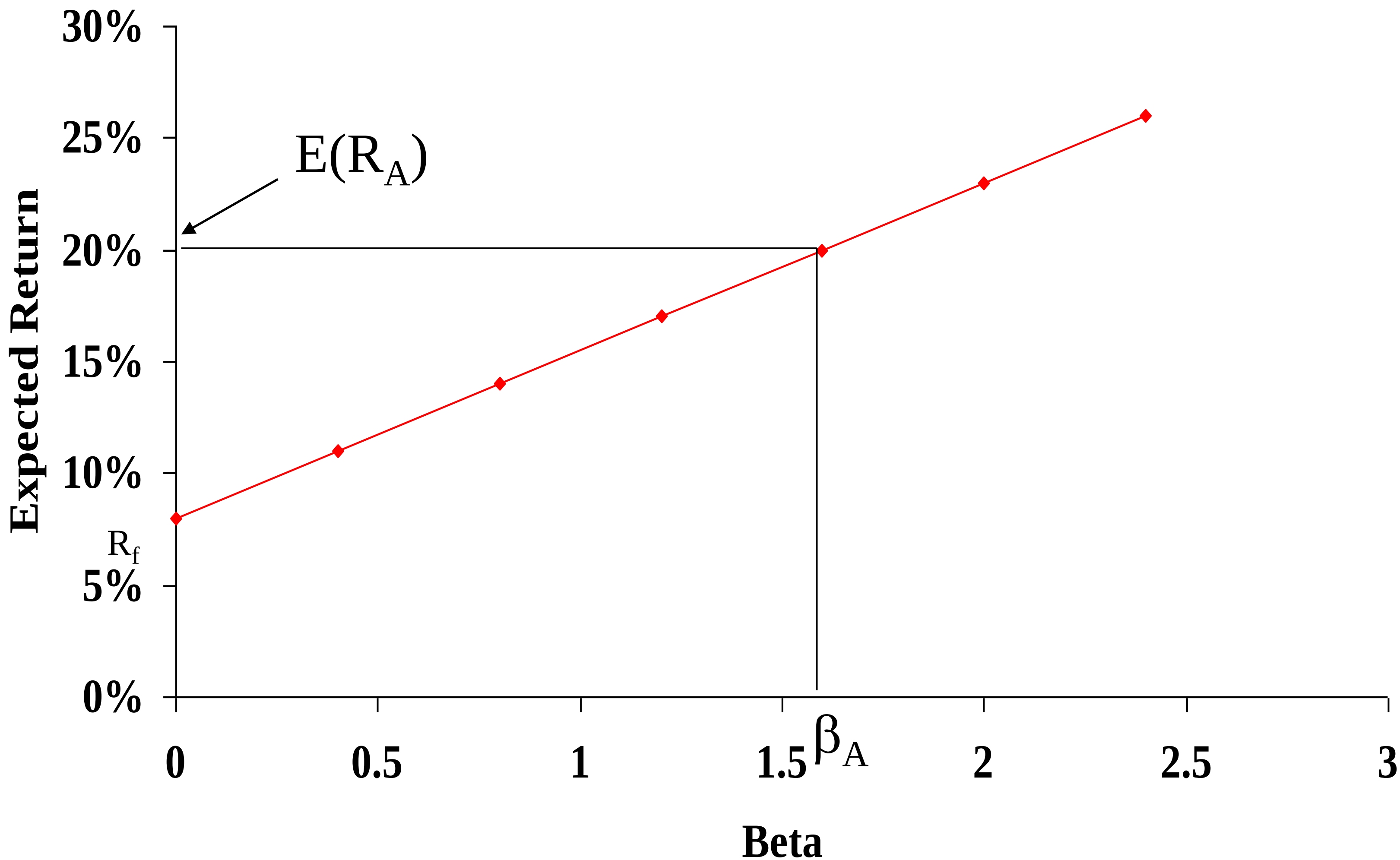
Security Market Line



Security Market Line

- The security market line (SML) is the representation of market equilibrium
- The slope of the SML is the reward-to-risk ratio: $(E(R_M) - R_f) / \beta_M$
- But since the beta for the market is ALWAYS equal to one, the slope can be rewritten
- Slope = $E(R_M) - R_f$ = ***market risk premium***

Example: Portfolio Expected Returns and Betas



Reward-to-Risk Ratio: Definition and Example

- The reward-to-risk ratio is the slope of the line illustrated in the previous example
  - Slope = $(E(R_A) – R_f) / (\beta_A – 0)$
  - Reward-to-risk ratio for previous example = $(20 – 8) / (1.6 – 0) = 7.5$
- What if an asset has a reward-to-risk ratio of 8 (implying that the asset plots above the line)?
- What if an asset has a reward-to-risk ratio of 7 (implying that the asset plots below the line)?

Market Equilibrium

- In equilibrium (BIG if), all assets and portfolios must have the same reward-to-risk ratio and they all must equal the reward-to-risk ratio for the market

$$\frac{E(R_A) - R_f}{\beta_A} = \frac{E(R_M - R_f)}{\beta_M}$$

Example - CAPM

- Consider the betas for each of the assets given earlier. If the risk-free rate is 4.5% and the market risk premium is 8.5%, what is the expected return for each?

Security	Beta	Expected Return
DCLK	3.69	$4.5 + 3.69(8.5) = 35.865\%$
KO	.64	$4.5 + .64(8.5) = 9.940\%$
INTC	1.64	$4.5 + 1.64(8.5) = 18.440\%$
KEI	1.79	$4.5 + 1.79(8.5) = 19.715\%$

Measuring Systematic Risk

- How do we measure systematic risk?
- We use the **beta coefficient to measure systematic risk**
- Beta measures a stock's market risk, and shows a stock's volatility relative to the market.
- Indicates how risky a stock is if the stock is held in a well-diversified portfolio.

What does beta tell us?

- A beta of 1 implies the asset has the same systematic risk as the overall market
- A beta < 1 implies the asset has less systematic risk than the overall market
- A beta > 1 implies the asset has more systematic risk than the overall market

Understanding Beta

Source: http://www.investopedia.com/video/

Total versus Systematic Risk

- Consider the following information:

	Standard Deviation	Beta
◦ Security C	20%	1.25
◦ Security K	30%	0.95

- Which security has more total risk? K
- Which security has more systematic risk? C
- Which security should have the higher expected return? C (why?)

Portfolio Beta

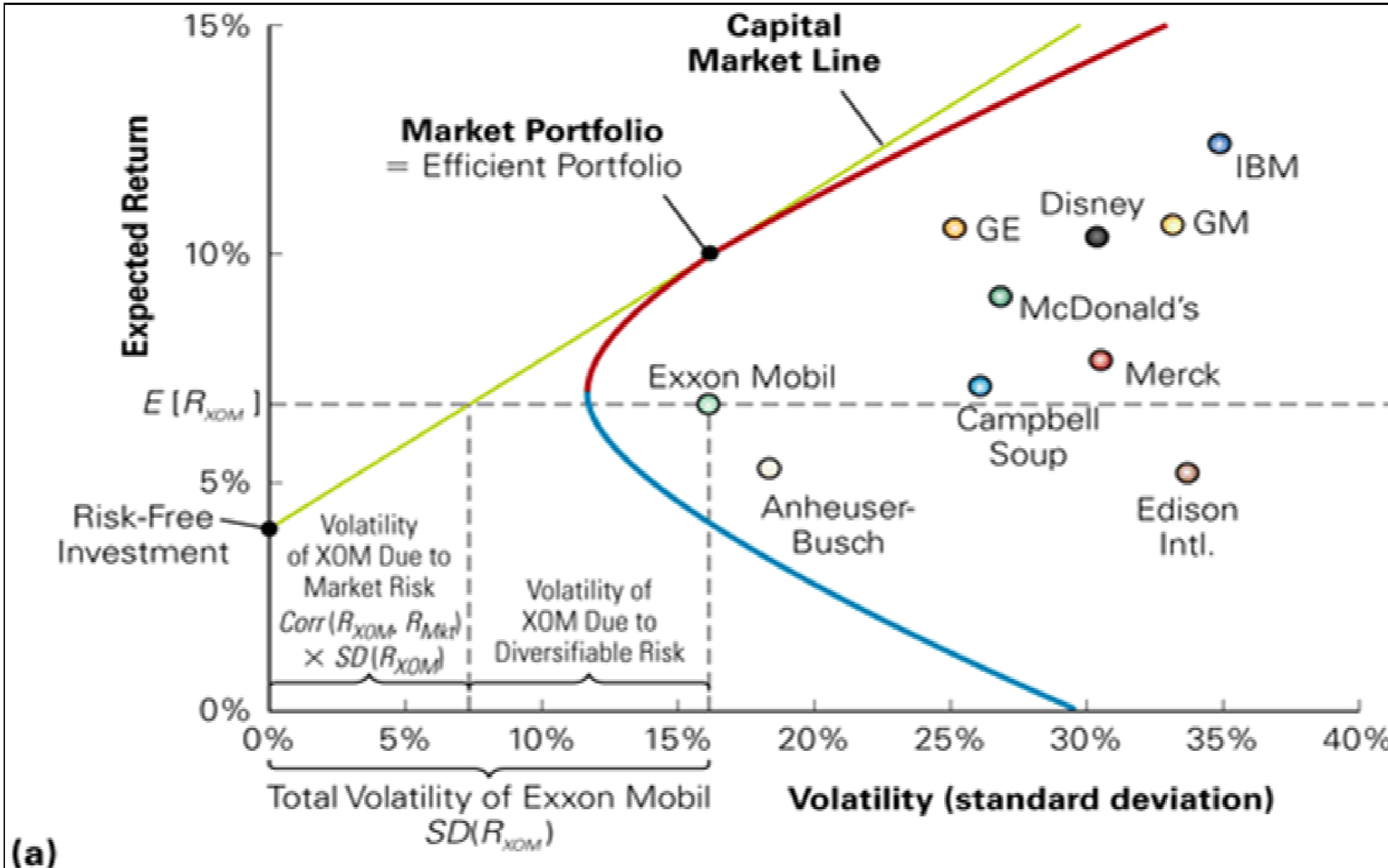
- Portfolio Beta is the value-weighted average of the betas of the assets in the portfolio: $\beta_p = x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n$
- Two-asset Example:

$$\begin{aligned}
\beta_p = \frac{\mathrm{Cov}(R_p, R_m)}{\mathrm{Var}(R_m)} &= \frac{\mathrm{Cov}(x_1 R_1 + x_2 R_2, R_m)}{\mathrm{Var}(R_m)} \\
&= \frac{\mathrm{Cov}(x_1 R_1, R_m) + \mathrm{Cov}(x_2 R_2, R_m)}{\mathrm{Var}(R_m)} \\
&= \frac{x_1 \mathrm{Cov}(R_1, R_m) + x_2 \mathrm{Cov}(R_2, R_m)}{\mathrm{Var}(R_m)} \\
&= \frac{x_1 \mathrm{Cov}(R_1, R_m)}{\mathrm{Var}(R_m)} + \frac{x_2 \mathrm{Cov}(R_2, R_m)}{\mathrm{Var}(R_m)} = x_1\beta_1 + x_2\beta_2
\end{aligned}$$

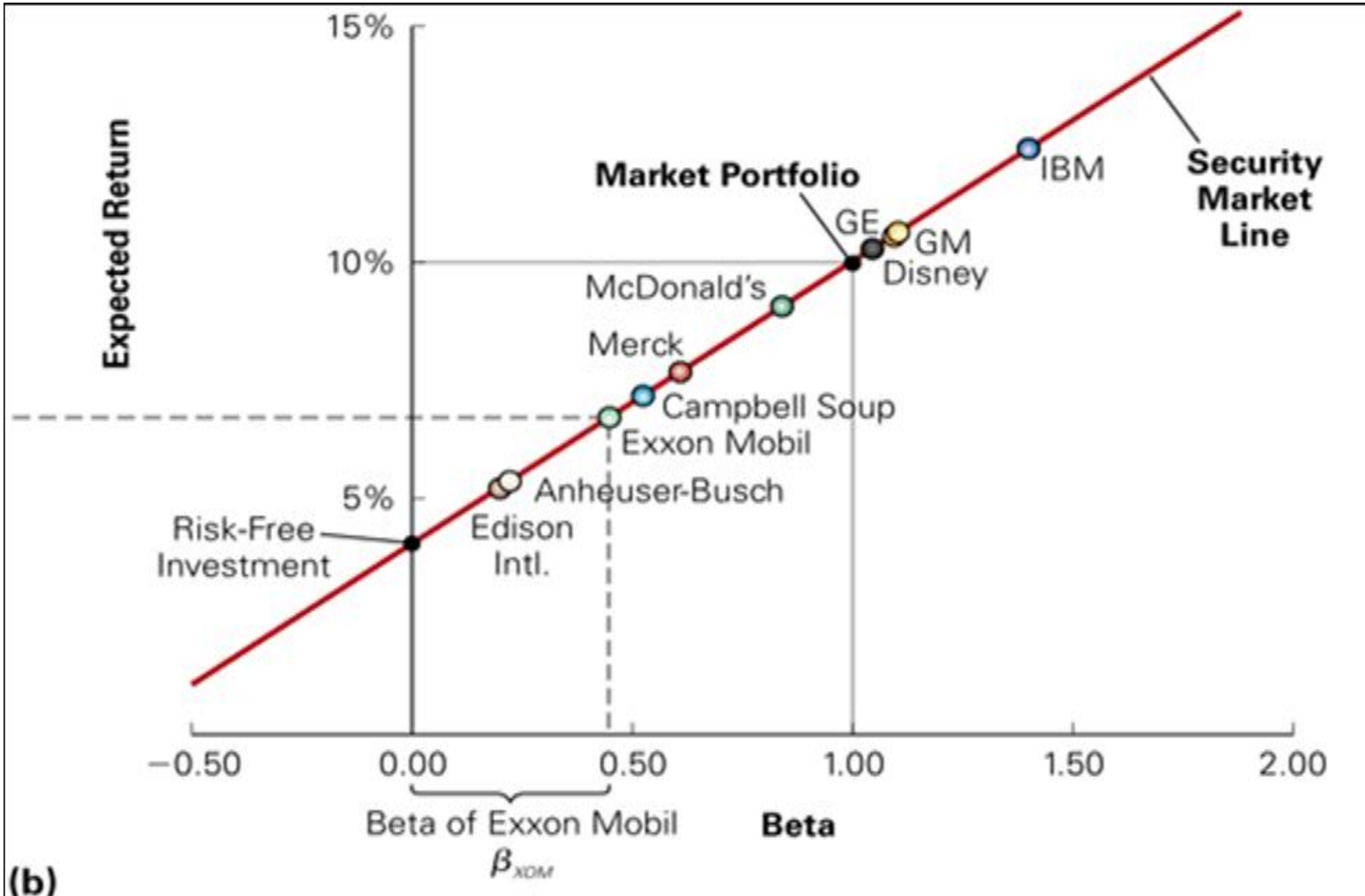
Example: Portfolio Betas

- Consider the previous example with the following four securities
Security	Weight	Beta
DCLK	.133	3.69
KO	.2	0.64
INTC	.267	1.64
KEI	.4	1.79
- What is the portfolio beta?
- .133(3.69) + .2(.64) + .267(1.64) + .4(1.79) = 1.77

Capital Market Line



Security Market Line



SML vs CML

CML	SML
Apply to efficient portfolio only. No Individual Security will lie on the CML	Apply to both Efficient Portfolios and Individual Stocks lie on SML
Plot risk premium (Efficient Portfolio) and portfolio standard deviation	Plot risk premium (Individual asset) on the asset's systematic risk
Use standard deviation as a measure of risk for efficient portfolio	Use beta, the contribution of the asset to the portfolio variance, as the measure of risk for individual assets held as part of a well-diversified portfolio
Illustrate the total risk (systematic and unsystematic)	Illustrate systematic risk only

Quick Quiz

- How do you compute the expected return and standard deviation for an individual asset? For a portfolio?
- What is the difference between systematic and unsystematic risk?
- What type of risk is relevant for determining the expected return?

End of Lesson