The Time Value of Money

Warren Buffett's Advice

- On spending, "If you buy things you do not need, soon you will have to sell things you need"
- On savings, "Do not save what is left after spending, but spend what is left after saving"
- On earnings, "Never depend on single income, make investment to create a second source"
- On investment, "Do no put all eggs in one basket"
- On taking risks, "Never test the depth of river with both the feet"

Lecture Outline

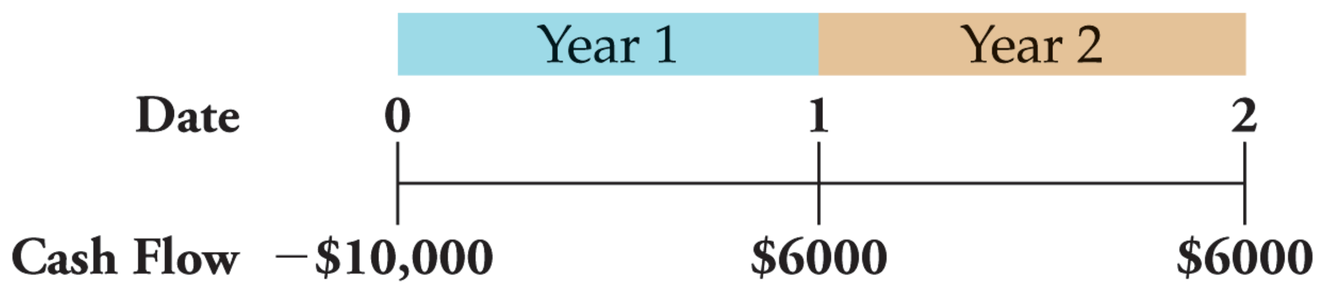
- Future Value and Compounding
- Present Value and Discounting
- Discount Rate
- Number of Periods
- Annuities and Perpetuities
- Formulas for Annuities and Perpetuities
- EAR and APR
- Interest Rates and Inflation

Basic Definitions

- Present Value – earlier money on a time line
- Future Value – later money on a time line
- Interest rate – "exchange rate" between earlier money and later money

The Timeline

- Assume that you are lending $10,000 today and that the loan will be repaid in two annual $6,000 payments.



Three Rules of Time Travel

TABLE 4.1 **The Three Rules of Time Travel**

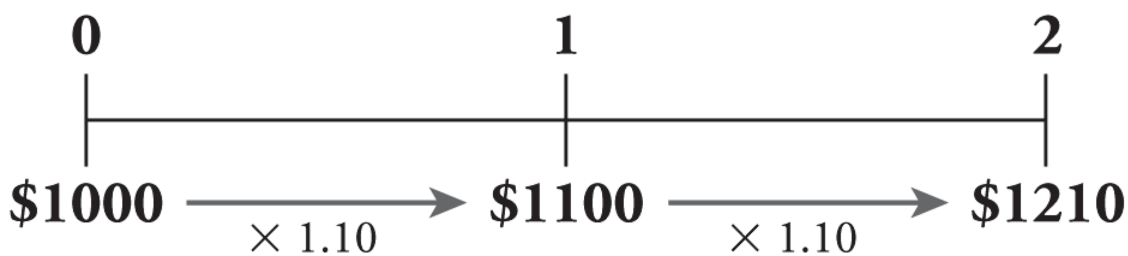
Rule 1	Only values at the same point in time can be compared or combined.	
Rule 2	To move a cash flow forward in time, you must compound it.	Future Value of a Cash Flow $FV_n = C \times (1 + r)^n$
Rule 3	To move a cash flow backward in time, you must discount it.	Present Value of a Cash Flow $PV = C \div (1 + r)^n = \frac{C}{(1 + r)^n}$

The 1st Rule of Time Travel

- A dollar today and a dollar in one year are not equivalent.
- It is only possible to compare or combine values at the same point in time.

The 2nd Rule of Time Travel

- To move a cash flow forward in time, you must compound it.



- Future Value of a Cash Flow

$$FV_n = C \times \underbrace{(1+r) \times (1+r) \times \cdots \times (1+r)}_{n \text{ times}} = C \times (1+r)^n$$

The 3rd Rule of Time Travel

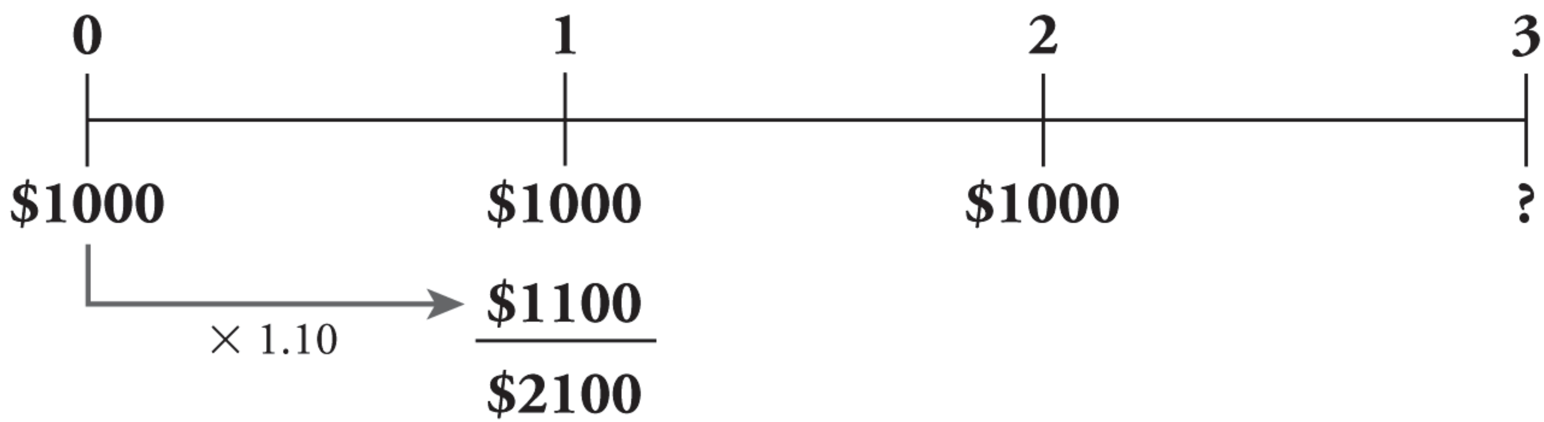
- To move a cash flow backward in time, we must discount it.
- Present Value of a Cash Flow

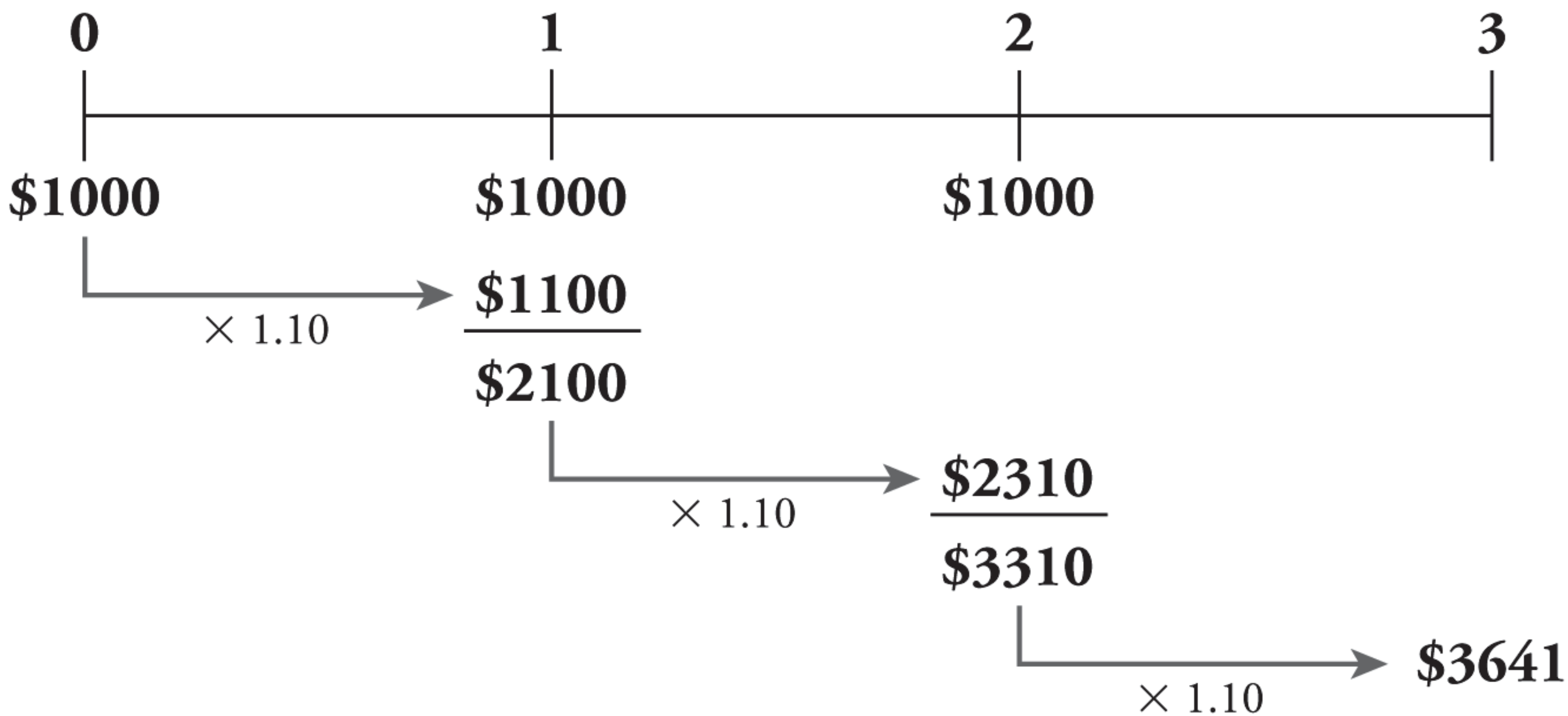
$$PV = C \div (1 + r)^n = \frac{C}{(1 + r)^n}$$

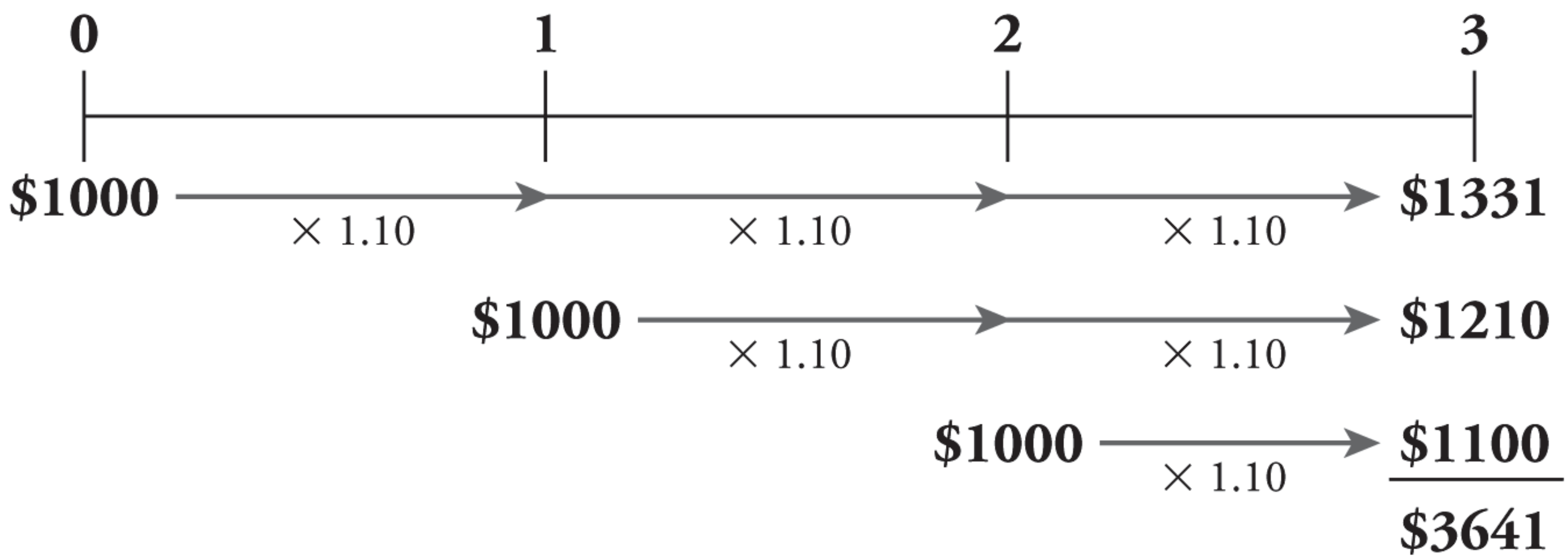
Combining Values Using the Rules of Time Travel

- Suppose we plan to save $1000 today, and $1000 at the end of each of the next two years. If we can earn a fixed 10% interest rate on our savings, how much will we have three years from today?

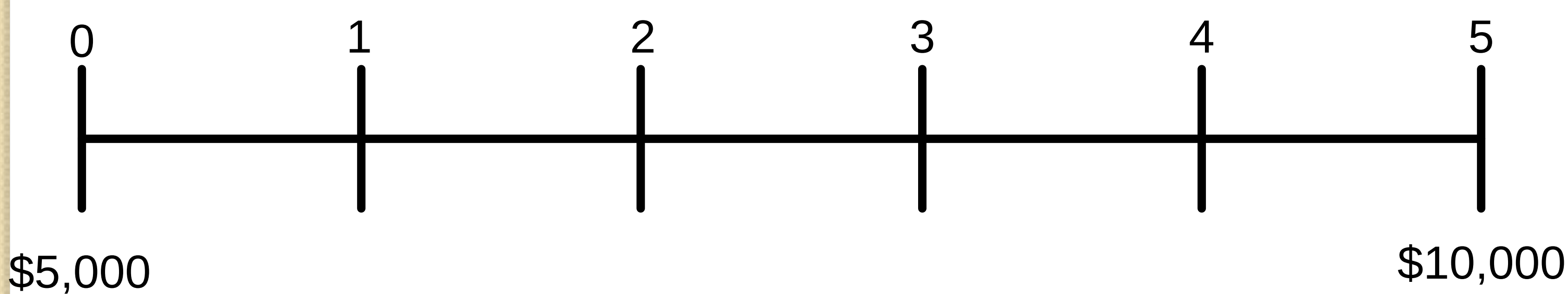
- The time line would look like this:


0
1
2
3
$1000
$1000
$1000
?
× 1.10
$1100
$2100

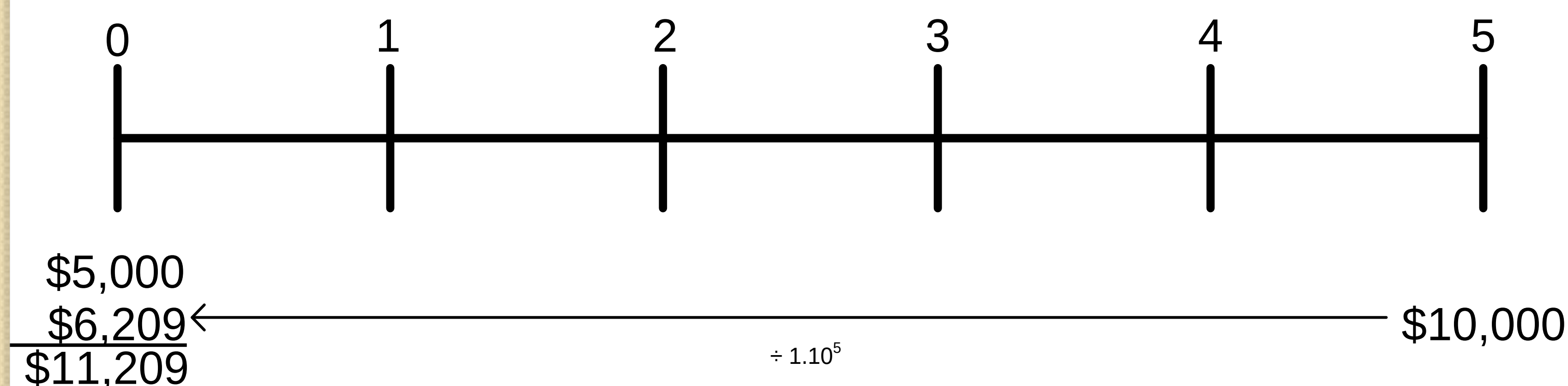

0
1
2
3
$1000
$1000
$1000
$1100
× 1.10
$2100
$2310
× 1.10
$3310
$3641
× 1.10


0
1
2
3
$1000
× 1.10
× 1.10
× 1.10
$1331
$1000
× 1.10
× 1.10
$1210
$1000
× 1.10
$1100
$3641

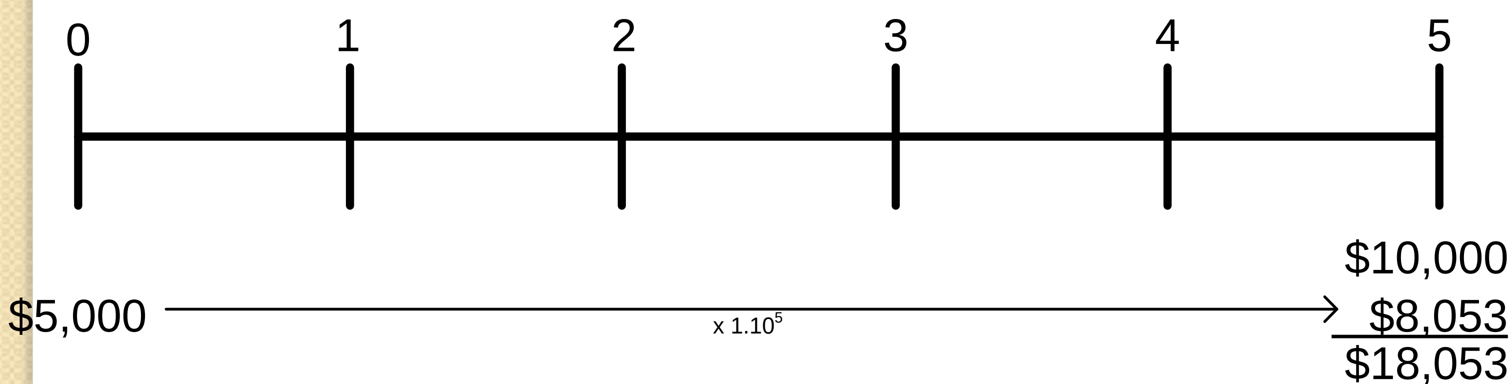
- Assume that an investment will pay you $5,000 now and $10,000 in five years.
  - The time line would like this:

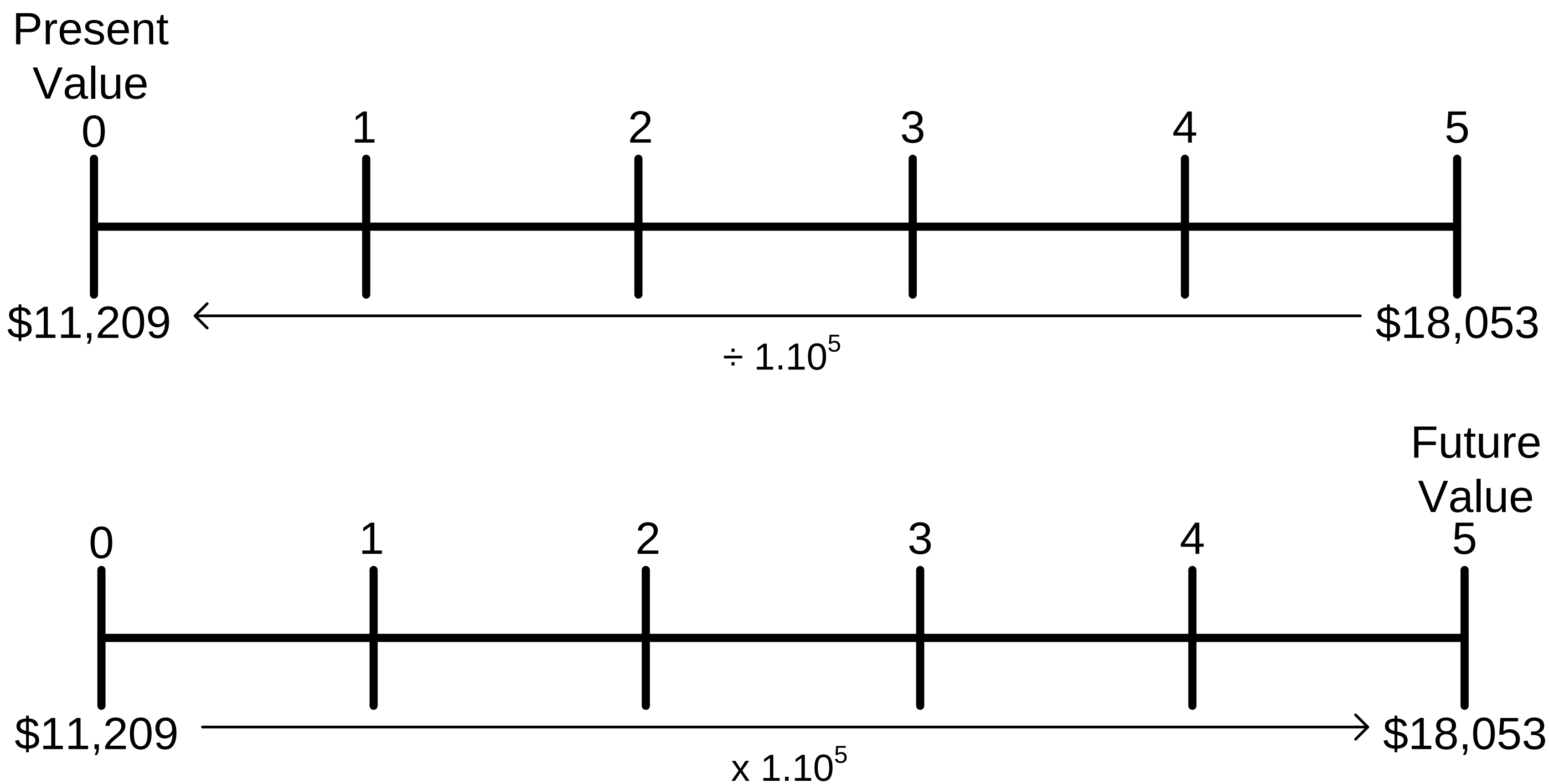


- You can calculate the present value of the combined cash flows by adding their values today.



- You can calculate the future value of the combined cash flows by adding their values in Year 5.

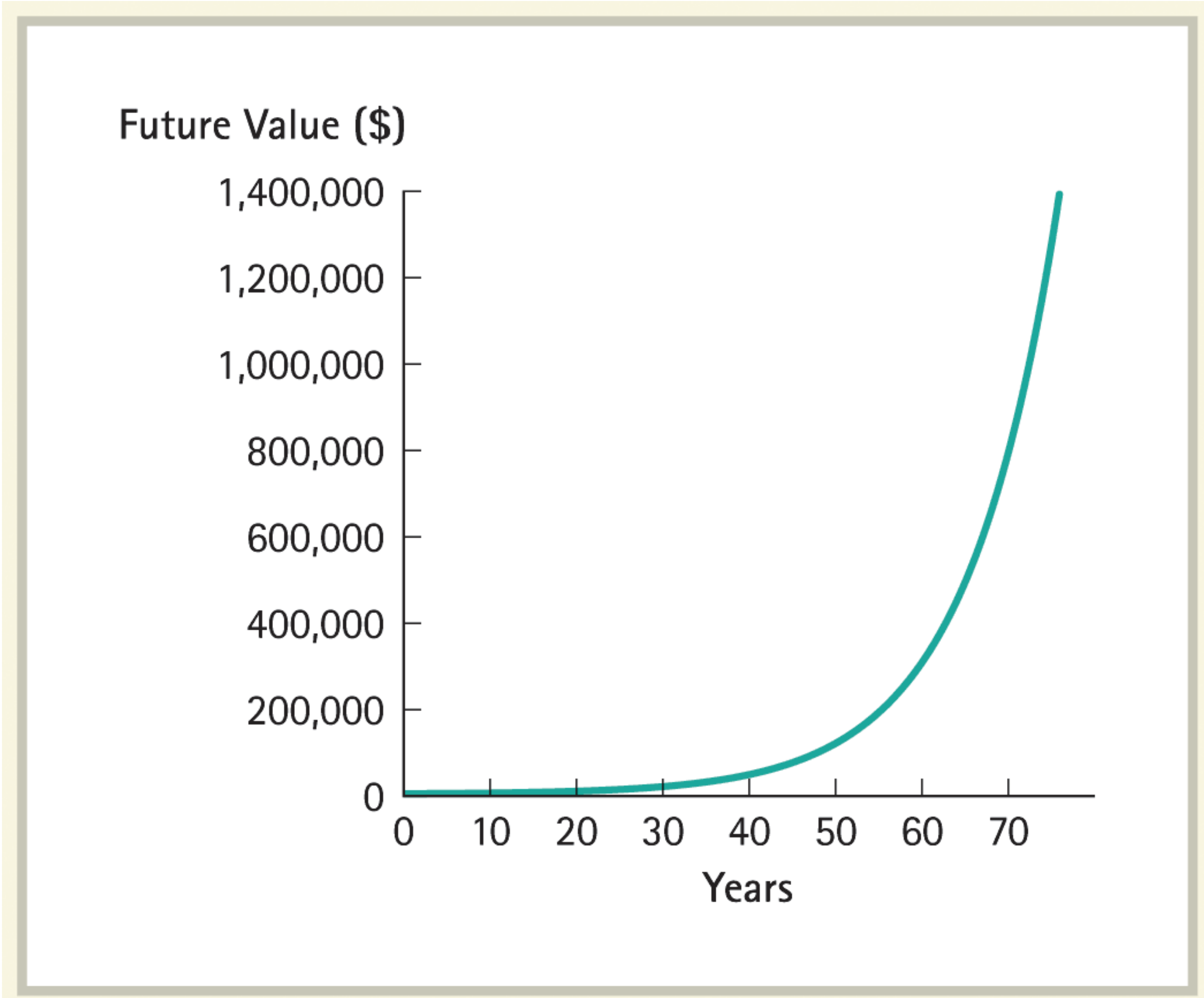



Present Value
0
1
2
3
4
5
$11,209
$18,053
÷ 1.10^5
Future Value
0
1
2
3
4
5
$11,209
$18,053
x 1.10^5

The Power of Compounding

- Compounding
  - Interest on Interest
    - As the number of time periods increases, the future value increases, at an increasing rate since there is more interest on interest.

The Power of Compounding

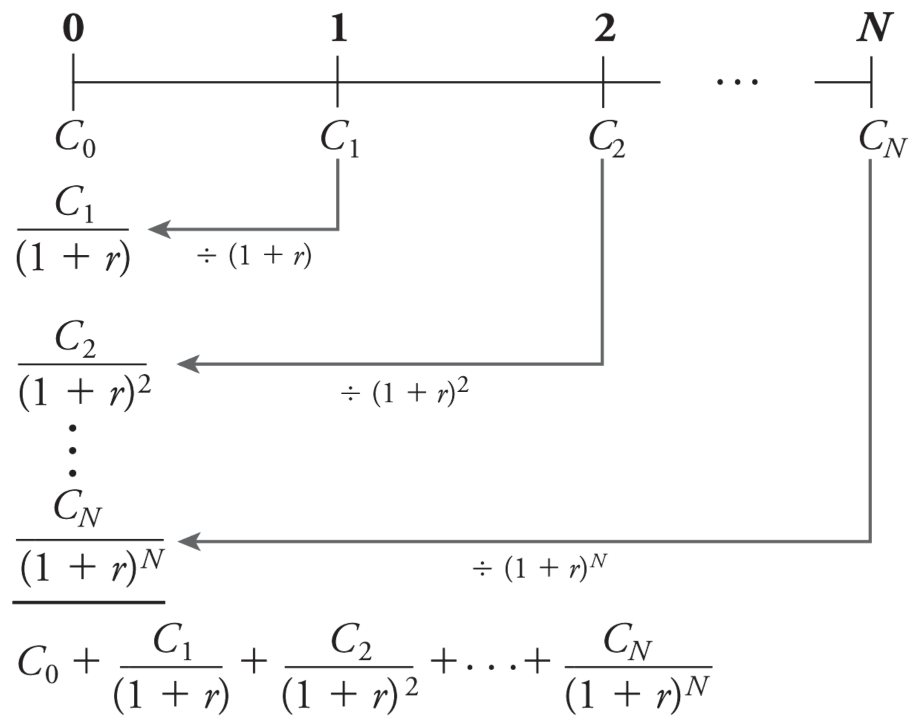


Rule of 72

Rate	Actual Years	Rule of 72
0.25%	277.605	288.000
0.5%	138.976	144.000
1%	69.661	72.000
2%	35.003	36.000
3%	23.450	24.000
4%	17.673	18.000
5%	14.207	14.400
6%	11.896	12.000
7%	10.245	10.286
8%	9.006	9.000
9%	8.043	8.000
10%	7.273	7.200
11%	6.642	6.545
12%	6.116	6.000
15%	4.959	4.800
18%	4.188	4.000
20%	3.802	3.600
25%	3.106	2.880
30%	2.642	2.400
40%	2.060	1.800
50%	1.710	1.440
60%	1.475	1.200
70%	1.306	1.029

Valuing a Stream of Cash Flows

- Based on the first rule of time travel we can derive a general formula for valuing a stream of cash flows: if we want to find the present value of a stream of cash flows, we simply add up the present values of each.



- Present Value of a Cash Flow Stream

$$PV = \sum_{n=0}^{N} PV(C_n) = \sum_{n=0}^{N} \frac{C_n}{(1+r)^n}$$

Example

Net Present Value of an Investment Opportunity

Problem

You have been offered the following investment opportunity: If you invest \$1000 today, you will receive \$500 at the end of each of the next three years. If you could otherwise earn 10% per year on your money, should you undertake the investment opportunity?

Solution

As always, start with a timeline. We denote the upfront investment as a negative cash flow (because it is money we need to spend) and the money we receive as a positive cash flow.

To decide whether we should accept this opportunity, we compute the NPV by computing the present value of the stream:

$$NPV = -1000 + \frac{500}{1.10} + \frac{500}{1.10^2} + \frac{500}{1.10^3} = \$243.43$$

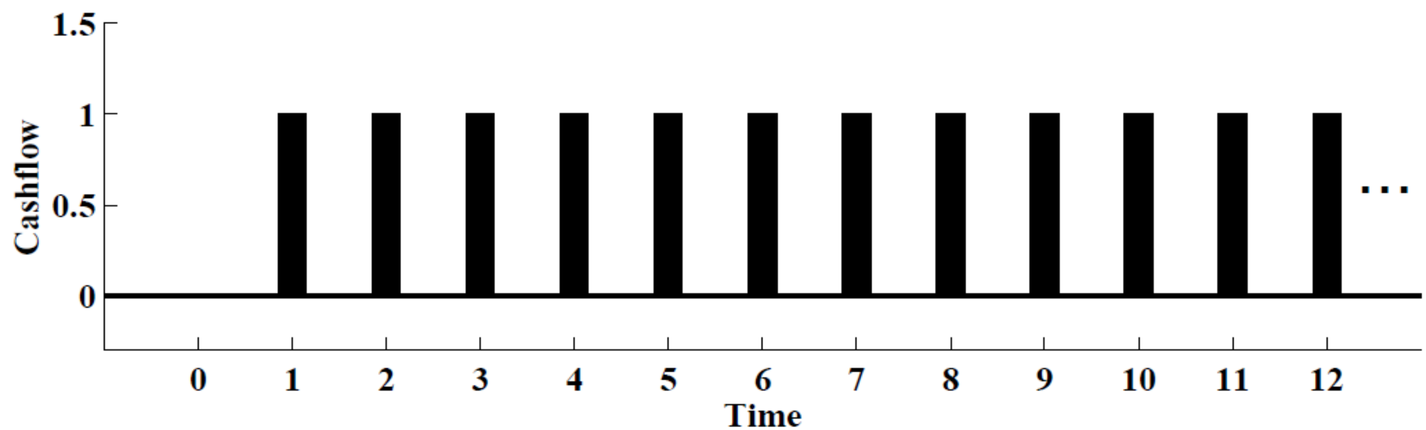
Because the NPV is positive, the benefits exceed the costs and we should make the investment. Indeed, the NPV tells us that taking this opportunity is like getting an extra $243.43

Annuities and Perpetuities Defined

- Annuity – finite series of equal payments that occur at regular intervals
  - If the first payment occurs at the end of the period, it is called an ordinary annuity
  - If the first payment occurs at the beginning of the period, it is called an annuity due
- Perpetuity – infinite series of equal payments

Valuing a Perpetuity

- A perpetuity is a constant stream of cash flows lasting forever.



- Mathematically,

$$P = C \times (1+r)^{-1} + C \times (1+r)^{-2} + C \times (1+r)^{-3} + \cdots$$

Valuing a Perpetuity

- Suppose you bought the perpetuity today and sold it for P_1 in a year. The PV of the perpetuity is the PV of the cash flow $C+P_1$

$$P = \frac{C + P_1}{1 + r}$$

- But the PV of the perpetuity must be the same at each point of time because the future cash flows are identical. So $P_1 = P$
- Thus, $P = \frac{C + P}{1 + r}$ or $\boxed{P = \frac{C}{r}}$

Alternatively…

- Mathematically, summing a geometric series that goes to infinity:

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \Leftrightarrow |r| < 1$$

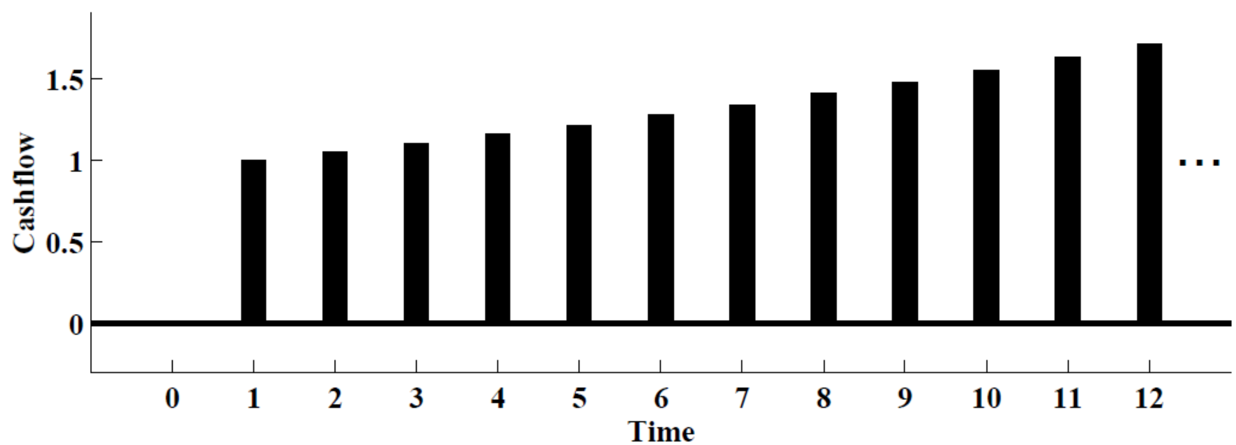
- Perpetuity is:

$$P = C \times (1+r)^{-1} + C \times (1+r)^{-2} + C \times (1+r)^{-3} + \cdots$$

- $A =$ $C \times (1+r)^{-1}$
- r $=$ $(1+r)^{-1}$
- What is P?

Value a Growing Perpetuity

- A growing perpetuity provides a cash flow of \$C in one year, and grows at a rate of g each subsequent year. The cash flows from a growing unit perpetuity looks like this (g=5%)



- Mathematically,

$$GP = C \times (1+r)^{-1} + C \times (1+g)(1+r)^{-2} + C \times (1+g)^2(1+r)^{-3} + \cdots$$

Valuing a Growing Perpetuity

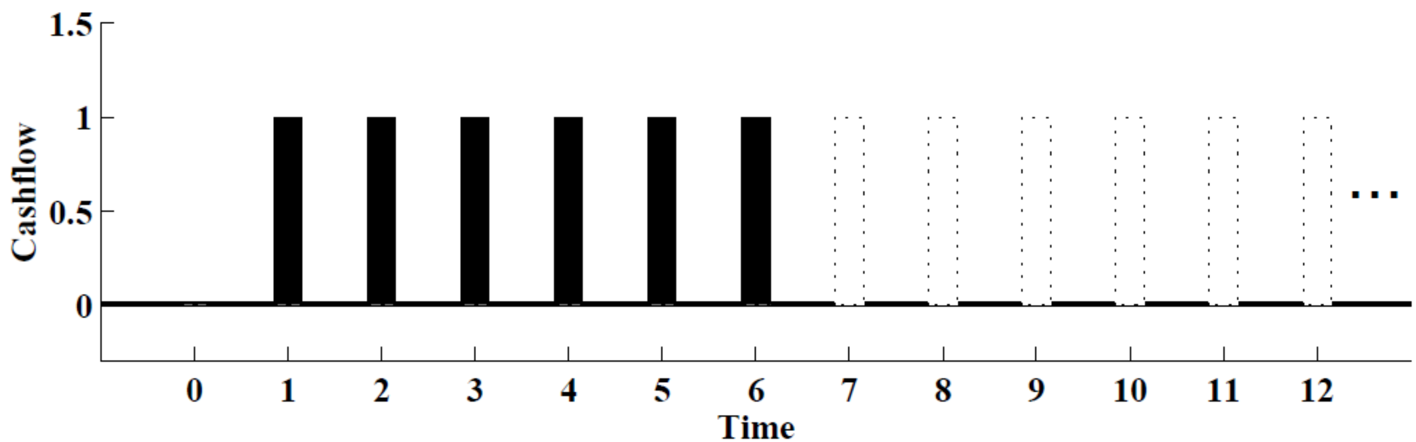
- Suppose you bought the growing perpetuity today and sold it for P_1 in a year. The PV of the perpetuity is the PV of the cash flow $C+P_1$

$$GP = \frac{C + P_1}{1 + r}$$

- The price in 1 year must be (1+g) times its price now, because its future cash flows are all g% larger. So, $P_1 = GP \times (1 + g)$
- Thus, $GP = \dfrac{C + GP(1 + g)}{1 + r}$ or $\boxed{GP = \frac{C}{r-g}}$

Valuing an Annuity

- An Annuity is a constant stream of cash flows lasting for a fixed number of years.



- The PV formula for an annuity is simply time 0 perpetuity minus time t perpetuity

Valuing an Annuity

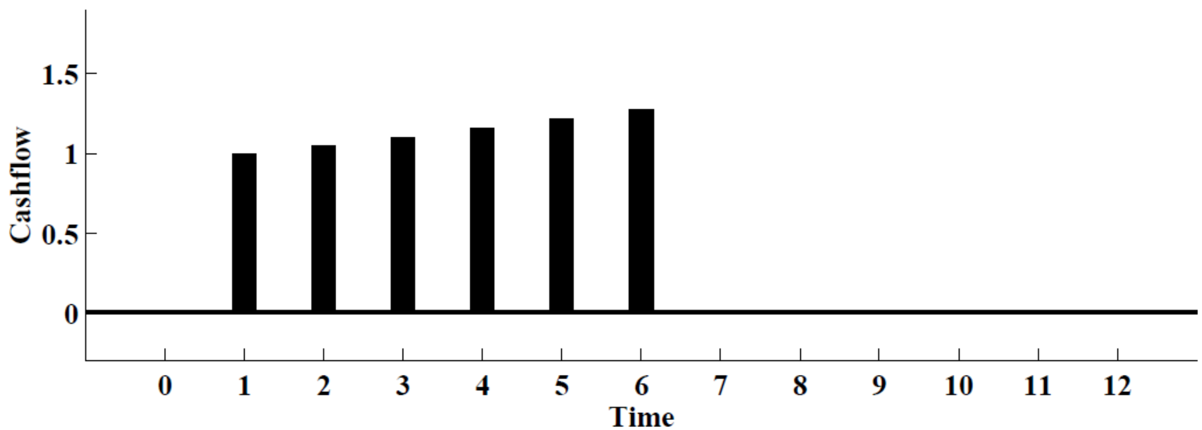
- The PV of time 0 perpetuity is $\frac{C}{r}$
- The PV of time t perpetuity is $\frac{1}{(1+r)^t} \cdot \frac{C}{r}$
- The value of the annuity is the difference:

$$\text{Ann} = \frac{C}{r} - \frac{1}{(1+r)^t} \cdot \frac{C}{r} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^t}\right)$$

- What if the first payment occurs today (year 0) rather than in a year?
- What is the future value (at time 6) of the annuity?

Valuing a Growing Annuity

- A growing annuity is a stream of cash flows that grows at a constant rate, say, g, for a fixed number of periods. The cash flows of a growing unit annuity look like this:



- Again, the value of the growing annuity is calculated by noting that it is a growing perpetuity at year 0 minus a growing perpetuity at year t

Valuing a Growing Annuity

- The PV of time 0 growing perpetuity is $\frac{C}{r-g}$
- The time t growing perpetuity is $\frac{C\cdot(1+g)^t}{r-g}$ and its PV is $\frac{C\cdot(1+g)^t}{r-g} \times \frac{1}{(1+r)^t}$
- The value of the annuity is the difference:

$$\mathrm{GAnn} = \frac{C}{r-g} - \frac{C}{r-g}\left(\frac{1+g}{1+r}\right)^t = \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^t\right]$$

Annuity (1) – Lottery Example

- Suppose you win $10 million lottery. The money is paid in equal annual installments of $333,333.33 over 30 years. If the appropriate discount rate is 5%, how much is the lottery actually worth today?

$$PV = C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

  - 30 N
  - 5 I/Y
  - 333,333.33 PMT
  - CPT PV = 5,124,150.29

Annuity (2)

- Suppose you deposit $50 a month into an account that has annual interest of 9%, based on monthly compounding. How much will you have in the account in 35 years?
  - Monthly rate = .09 / 12 = .0075
  - Number of months = 35(12) = 420

  - 35(12) = 420 N
  - 9 / 12 = .75 I/Y
  - 50 PMT
  - CPT FV = 147,089.22

Finding the Payment (1)

- Suppose you want to borrow $20,000 for a new car. You can borrow at 8% per year, ***compounded monthly*** (8/12 = .66667% per month). If you take a 4 year loan, what is your monthly payment?

$$\mathrm{PV} = \mathrm{C} \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

  - 4(12) = 48 N
  - 20,000 PV
  - 0.66667 I/Y
  - CPT PMT = -488.26

Finding the Payment (2)

- A pension fund manager must fund a \$10M obligation due in 10 years. What annual contributions are needed? Suppose r = 5%.

$$\mathrm{FV} = C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right] \times (1+r)^t$$

  - 10 N
  - 10,000,000 FV
  - 5 I/Y
  - CPT PMT = -795,046

Finding the Number of Payments (1)

- An entrepreneur borrows \$300,000 today. The interest rate is 8%. If the entrepreneur makes annual payments of \$45,000 per year how many years will it take to repay the loan?

$$PV = C \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^{t}}\right]$$

  - 8 I/Y
  - 300,000 PV
  - -45,000 PMT
  - CPT N = 9.9

Finding the Rate

- Suppose you borrow $10,000 from your parents to buy a car. You agree to pay $207.58 per month for 60 months. What is the monthly interest rate?

$$\mathrm{PV} = \mathrm{C} \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right]$$

  - Sign convention matters!
  - 60 N
  - 10,000 PV
  - -207.58 PMT
  - CPT I/Y = .75%

Future Values for Annuities

- Suppose you begin saving for your retirement by depositing $2000 per year in an IRA. If the interest rate is 7.5%, how much will you have in 40 years?

$$\mathrm{FV} = \mathrm{C} \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^t}\right] \times (1+r)^t$$

  - 40 N
  - 7.5 I/Y
  - -2000 PMT
  - CPT FV = 454.513.04

Perpetuity

- Company A has an existing perpetual bond that pays $1 quarterly, which is currently priced at $40. If the company plans to raise $100 by issuing a new perpetual bond that pays quarterly, how much dividend per quarter should the company pays?

$$P = \frac{C}{r}$$

- Current required return:
  - 40 = 1 / r
  - r = .025 or 2.5% per quarter
- Dividend for new perpetual bond:
  - 100 = C / .025
  - C = 2.50 per quarter

Growing Perpetuity

- A company will pay an annual dividend next year of $3. You expect its dividend to grow at the rate of 5% per year forever. You calculate that the expected return on their equity should be 10%. What should be the company's price per share?

$$\mathrm{PV} = \frac{3}{1.10} + \frac{3(1+0.05)}{1.10^2} + \frac{3(1+0.05)^2}{1.10^3} + \cdots$$

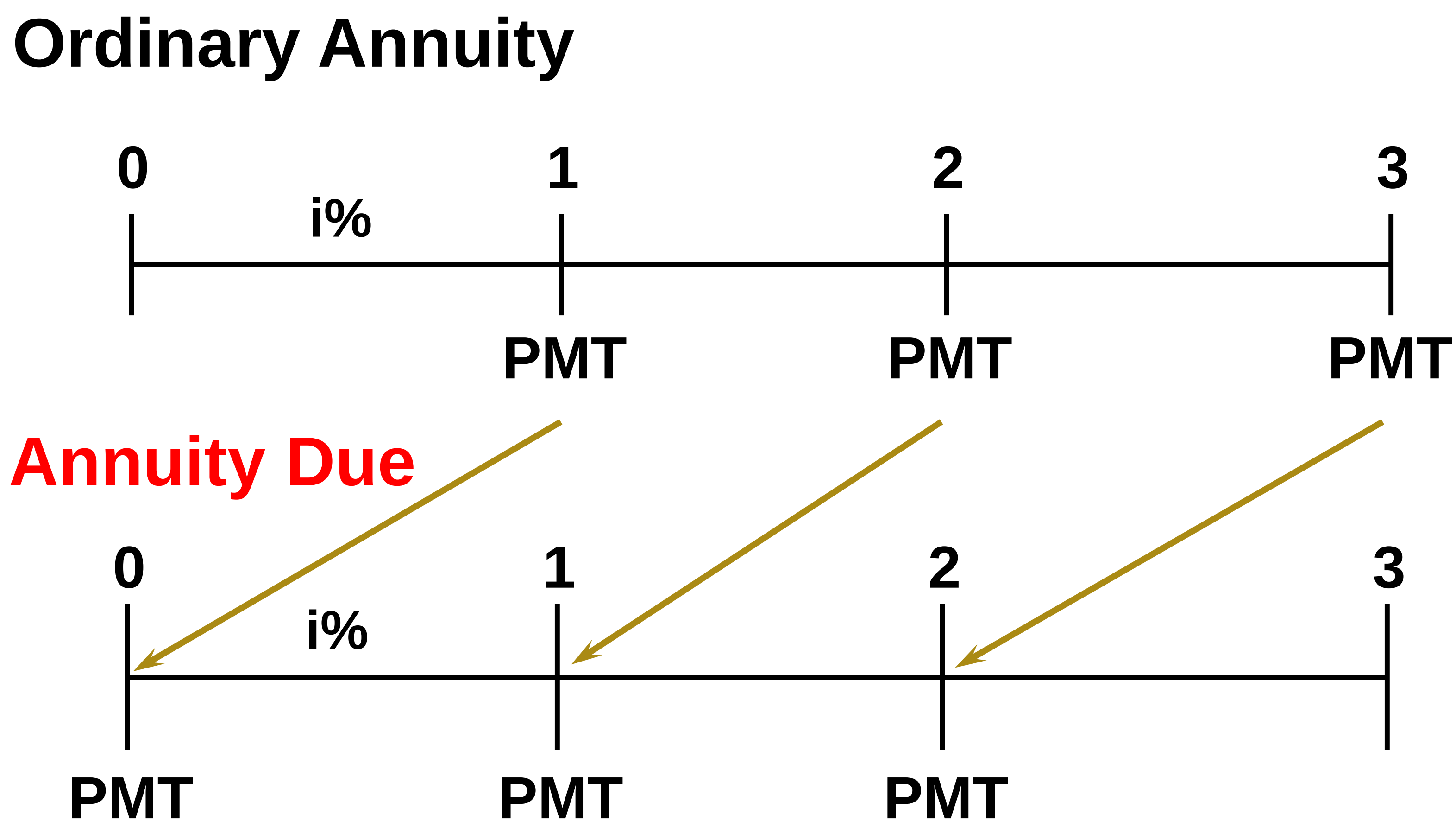
$$= \frac{3}{0.10 - 0.05}$$

$$= 60$$

Important Notes

- The previous examples are direct applications of the annuity/perpetuity formula.
- In trickier scenarios, we need to be careful about the timing of the cash flows.
  - When do cash flows occur?
  - How often do they occur?

What is the difference between an ordinary annuity and an annuity due?



Annuity Due (I) – Lottery Example

- Bob has just won the lottery, paying 20 equal installments of $50,000 each. He receives his first payment now, i.e. at year 0. If the interest rate is 8 percent, what is the present value of the lottery?

$$\begin{aligned} \mathsf{PV}_{t=0} &= 50000 + \mathsf{PV}_{t=0}(19 \text{ cash flows}) \\ &= 50000 + \frac{50000}{0.08}\left(1 - \frac{1}{(1+0.08)^{19}}\right) = 530180 \end{aligned}$$

OR

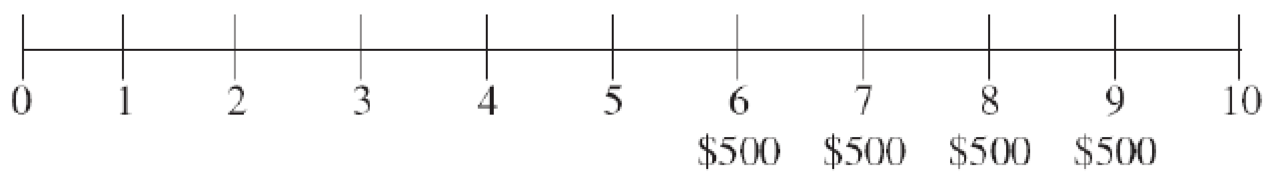
$$\begin{aligned} \mathsf{PV}_{t=0} &= \mathsf{PV}_{t=-1}(20 \text{ cash flows}) \times (1+r) \\ &= \frac{50000}{0.08}\left(1 - \frac{1}{(1+0.08)^{20}}\right) \times 1.08 = 530180 \end{aligned}$$

Annuity Due (2)

- You are saving for a new house and you put $10,000 per year in an account paying 8%. The first payment is made today. How much will you have at the end of 3 years?
  - Set Type=1
  - 3 N
  - -10,000 PMT
  - 8 I/Y
  - CPT FV = 35,061.12

Delayed Annuity - Saving For Retirement Example

- You are offered the opportunity to put some money away for retirement. You will receive four annual payments of \$500 each beginning at year 6. How much would you be willing to invest today if you desire an interest rate of 10%?



$$PV_{t=5} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^t}\right) = \frac{500}{0.1}\left(1 - \frac{1}{(1.1)^4}\right) = 1585$$

$$PV_{t=0} = PV_{t=5} \times \frac{1}{(1+r)^5} = 1585 \times \frac{1}{1.1^5} = 984$$

An Infrequent Annuity

- Charlie receives an annuity of \$450, payable once every two years. The annuity stretches out over 20 years. The first payment occurs at date 2, that is, two years from today. The annual interest rate is 6 percent. What is the present value of the annuity?
- The (annual) interest rate over two-year period, denoted by R, is

$$(1+R) = (1+r)^2 \quad \Rightarrow \quad R = 0.1236$$

- So, $\mathrm{PV} = \frac{450}{R}\left(1 - \frac{1}{(1+R)^{10}}\right) = 2505.57$

Multiple Annuities

- An insurance agent approaches Debra and would like to sell her the following contract: (1) Debra pays $5,000 per year for the coming 15 years. (2) In return, she will receive $7,000 a year for the following 15 years.
- Assume that interest rates will remain at a constant 9%. How much profit does the insurance company make? In order for the deal to generate zero profits, what an annual payment does Debra need to receive?

Multiple Annuities (con't)

- The insurance company's profit:

$$\text{Profit} = \frac{5000}{r}\left(1 - \frac{1}{(1+r)^{15}}\right) - \frac{7000}{r}\left(1 - \frac{1}{(1+r)^{15}}\right) \times \frac{1}{(1+r)^{15}}$$
$$= 40303.44 - 15490.76 = 24812.68$$

- For zero profit, we need:

$$\frac{5000}{r}\left(1 - \frac{1}{(1+r)^{15}}\right) = \frac{C}{r}\left(1 - \frac{1}{(1+r)^{15}}\right) \times \frac{1}{(1+r)^{15}}$$
$$\text{or } 5000 = \frac{C}{(1+r)^{15}} \Rightarrow C = 18212.41$$

Decisions, Decisions

- Your broker calls you and tells you that he has this great investment opportunity. If you invest $100 today, you will receive $40 in one year and $75 in two years. If you require a 15% return on investments of this risk, should you take the investment?
  - CF_0 = 0; C01 = 40; C02 = 75
  - I = 15
  - CPT NPV = 91.49
  - No – the broker is charging more than you would be willing to pay.

Summary

- Perpetuity
  - A constant stream of cash flows that lasts forever.
- Growing perpetuity
  - A stream of cash flows that grows at a constant rate forever.
- Annuity
  - A stream of constant cash flows that lasts for a fixed number of periods.
- Growing annuity
  - A stream of cash flows that grows at a constant rate for a fixed number of periods.

Summary

- We presented four simplifying formulae:

$$\text{Perpetuity} \;: PV = \frac{C}{r}$$

$$\text{Growing Perpetuity} \;: PV = \frac{C}{r-g}$$

$$\text{Annuity} \;: PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$$

$$\text{Growing Annuity} \;: PV = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{(1+r)}\right)^T\right]$$