Scenario Simulation Using Principal Components*

Key words: principal components analysis, Monte Carlo simulation, risk analysis, term structure of interest rates, forward curve

Portfolio managers, hedge funds, and derivative traders, among others, are confronted with the valuation and risk management of financial portfolios involving an increasingly large number of securities. The heterogeneity of asset pools makes these tasks very challenging and arduous to deal with using standard valuation tools, such as naive Monte Carlo methods.

Principal Components Analysis (PCA) and scenario simulation are complexity reduction techniques that can be applied to portfolios involving correlated assets.

The PCA method, as introduced in Steeley (1990) for the purpose of financial applications, consists of identifying statistically independent factors affecting the random evolution of a given pool of assets, sorting them by order of relative importance, and selecting the most significant ones. This task can be undertaken through the computation of a historical cross-asset covariance matrix and the identification of principal components representing independent factors underlying the market risk borne by the portfolio.

Jolliffe (1986) is the standard reference for PCA. Avellaneda and Scherer (2002) and D’Ecclesia and Zenios (1994) apply this technique to interest rate markets and portfolio selection. Gabbi and Sironi (2005) provide as extensive factor analysis in the Eurobond market. PCA has been applied to measure Value-at-Risk figures in Roncoroni (2004).¹ Based on early results by Roncoroni (1997, 1999), a functional version of PCA has been proposed by Guiotto and Roncoroni (2001), Roncoroni, Galluccio and Guiotto (2003), and Galluccio and Roncoroni (2006). These authors develop a method allowing to detect “Shape Factors” affecting cross-sectional de-

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¹ Gatti et al. (2006) and Maspero and Saita (2005) provide key applications of Value-at-Risk to asset management and project finance transactions.
formations and perform a study on the empirical performance of hedging strategies based on these factors compared to the traditional PCA components.

The scenario simulation method, as introduced in Jamshidian and Zhu (1997), consists of replacing the joint probability distributions of the significant factors selected using PCA by appropriate discrete-valued approximations and then evaluating the distribution of the market value of an asset portfolio as a function of these approximated distributions.

The main hypothesis underlying these methods is that factors are jointly normally distributed. Correspondingly, the performance of the resulting reduced-form model depends on four elements, namely

(1) the similarity of actual asset price distributions to normal variates,
(2) the correlation level among assets in the portfolio,
(3) the number of retained factors,
(4) the number of points in the approximating discrete distribution.

As these values become larger, the quality of the resulting model improves.

This case-study is organized as follows. Here the factor detection problem and the complexity reduction issue are introduced. Section 23.1 details the solution methodology from a theoretical viewpoint, and provides an application to interest rate markets. Section 23.2 describes an implementation algorithm and the corresponding code. Section 23.3 concludes with some numerical experiments aimed at testing the quality of the method.

### 23.1 Problem Statement and Solution Methodology

Simulating cross-sectional data, such as a term structure of interest rates, a commodity forward price curve, or a volatility smile, is a challenging task due to the high number of variables involved in the process. PCA aims at identifying and reducing the number of mutually independent factors driving the random evolution of a given cross-section. Scenario simulation improves PCA results by reducing the number of possible outcomes that need to be simulated for each selected factor.

We introduce the main issue through a simple example. Consider a portfolio $\pi$ involving interest-rate dependent securities only. The portfolio value $V_{\pi}$ depends on the term structure of interest rate expressed as a continuously compounded yield curve $y_x$, where $x \geq 0$ denotes a time-to-maturity. This can be seen as a prototypical instance of a cross-section. We assume that $\pi$ contains positions depending on a number $n$ of yields and that each yield may take one among a set of $k$ values within the portfolio time horizon. Evaluating the risk underlying the portfolio value by Monte Carlo requires several simulations of possible scenarios for the $n$ yields. In general, there are $k^n$ possible outcomes, though a reasonably well-approximated portfolio distribution can be obtained by fewer Monte Carlo simulations. However, the computational burden remains quite important even for small portfolios.
There are two sources of reducible complexity in this procedure. First, different yields may be correlated to a certain extent, so simulating their outcomes independently of each other introduces a loss of efficiency. PCA proposes a solution to this issue by replacing the $n$ yields with a number $m \ll n$ of mutually independent factors $w_1, \ldots, w_m$. Second, sampling the distribution of $V_\pi$ by simulating samples from factor distributions is highly time consuming due to the high number of possible states taken by each factor. For instance, a four-factor model ($m = 4$) with $M = 100$ states for each factor leads to $N = 100^4$ scenarios and a same number of sample portfolio values $V_\pi$. Monte Carlo simulation usually requires a high number, though smaller than $N$, of runs to come up with a reasonable assessment of the portfolio distribution. Scenario simulation suggests replacing the underlying distributions by discrete ones involving a very small number of states, say from 3 to 9 in most applications. Let us explore the two techniques to a greater depth.

Empirical analysis of interest rate dynamics shows that the yield curve $y = \{y_x, x \geq 0\}$ evolves over time according to a wide variety of possible deformations. For instance, Central Bank may decide to decrease the short-term rate while the market is quoting a higher 10-year bond yield; as a result, the yield curve is expected to steepen. This phenomenon can be described as follows.

We consider $n$ points on the yield curve. On a given time interval, all points may vary quite heterogeneously, giving the impression that $n$ factors are actually driving the yield curve evolution. A moment’s reflection suggests that pairs of yields corresponding to neighboring times-to-maturity are likely to move together whereas far-away yields are likely to have a lower degree of mutual dependence in their behavior. Qualitatively, we may say that a perturbation affecting a single yield spreads over the entire term structure according to the degree of proximity to the shocked point. This observation suggests that PCA should be performed on the empirical covariance (or correlation) matrix of absolute (or relative) yield returns.

Slightly relaxing our notation, we let $y_i(t)$ denote the time $t$ quoted yield corresponding to time-to-maturity $x_i$ ($i = 1, \ldots, n$). In a standard diffusion setting, we may assume yield return dynamics

\[
\frac{dy_1(t)}{y_1(t)} = \mu_1 \, dt + dW_1(t),
\]
\[
\frac{dy_2(t)}{y_2(t)} = \mu_2 \, dt + dW_2(t),
\]
\[
\vdots
\]
\[
\frac{dy_n(t)}{y_n(t)} = \mu_n \, dt + dW_n(t),
\]

(23.1)

where the $W_i$’s are correlated Brownian motions. Principal components analysis delivers dynamics for $y_i$’s in terms of independent Brownian shocks $\tilde{W}_i(t)$ ($i = 1, \ldots, n$). These processes are obtained as linear combinations of the original Brownian motion in that $d\tilde{W}_i = \sum_{k=1}^n \alpha_k \, dW_k$ for suitable coefficients $\alpha_1, \ldots, \alpha_n$. Moreover, they are ranked according to their relative importance in reproducing the
underlying volatility. Indeed, if \( \lambda_k \) denotes the instantaneous variance per time unit of the process \( \tilde{W}_k \), namely \( \lambda_k \, dt = \text{Var}(d\tilde{W}_i) \), then \( \lambda_i \geq \lambda_j \), whenever \( i > j \).

Complexity reduction can be achieved by setting a number \( \rho \in (0, 1] \) representing the proportion of the overall market volatility that needs to be reproduced by a reduced-form yield curve model and defining \( m \) as the smallest number of Brownian motions required for generating this figure, i.e.,

\[
m := \min \left\{ i \geq 1 : \frac{\sum_{k=1}^{i} \lambda_k}{\sum_{k=1}^{n} \lambda_k} \geq \rho \right\}.
\]

We then consider reduced-form dynamics

\[
\frac{dy_1(t)}{y_1(t)} = \mu_1 \, dt + \sum_{k=1}^{m} \beta_{1k} d\tilde{W}_k(t),
\]

\[
\frac{dy_2(t)}{y_2(t)} = \mu_2 \, dt + \sum_{k=1}^{m} \beta_{2k} d\tilde{W}_k(t),
\]

\[
\vdots
\]

\[
\frac{dy_n(t)}{y_n(t)} = \mu_n \, dt + \sum_{k=1}^{m} \beta_{nk} d\tilde{W}_k(t).
\] (23.2)

Once this representation is established, a scenario simulation can be performed on these dynamics. In most applications \( m \) is about \( 8-10 \) times lower than \( n \).

Standard Monte Carlo simulation requires sampling among hundreds of possible outcomes for the underlying Brownian motions. Scenario simulation is a reduction technique aimed at simplifying the burden of generating paths from the exact noise distribution. The idea is to select a very limited number of key states (called scenarios) of the noise driver, to assign them probabilities consistent with the initial probability distribution, and finally to sample from this reduced-form model.

Jamshidian and Zhu (1997) suggest selecting 9 states for the first factor, 7 for the second, 5 for the third, and 3 for the last driver. The scenario probability distribution is taken as a multinomial approximation of the underlying multivariate normal distribution. If \( m \) denotes the number of possible outcomes for each of the four factors, then the probability of a state \( i \) is given by

\[
P(i) = 2^{-m} \frac{m!}{i!(m-i)!}.
\]

The distance between two different states is \( 2/\sqrt{m} \) standard deviations and the distance between the center of the distribution and the farthest state is \( \frac{1}{2}m \frac{2}{\sqrt{m}} = \sqrt{m} \).

\section*{23.2 Implementation and Algorithm}

\subsection*{23.2.1 Principal Components Analysis}

Consider a vector-valued diffusion process with constant coefficients
\[
\frac{dy(t)}{y(t)} = \mu \, dt + dW(t),
\] (23.3)

where \( W \) is a correlated \( n \)-dimensional Brownian motion. Here division between vectors has to be interpreted componentwise, e.g., \( \frac{\mathbf{a}}{\mathbf{b}} = (\frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n}) \). A time series of cross-sectional data will be denoted by \( \{y(t), t = \delta, \ldots, N\delta\} \), where \( y(t) = (y_1(t), \ldots, y_n(t)) \).

**Algorithm (Principal Components Analysis on a Diffusion Process)**

1. **Data setting.** Consider the annualized relative yield variations \( \frac{y(t + \delta) - y(t)}{\delta y(t)} \) for all dates \( t = \delta, \ldots, N\delta. \) The sample mean of these data is given by

\[
\mu := (\delta N)^{-1} \sum_{k=1}^{N} \left[ \frac{y(k\delta) - y((k-1)\delta)}{y((k-1)\delta)} \right].
\] (23.4)

If we subtract this average from each variation observed in the market, we obtain a set of annualized centered relative yield variations

\[
\Delta(t) := \delta^{-1} \left[ \frac{y(t + \delta) - y(t)}{y(t)} \right] - \mu, \quad t = \delta, \ldots, N\delta.
\] (23.5)

The main hypothesis at this stage is that these vectors are all independent samples from a common multivariate normal distribution.

2. **Descriptive statistical analysis.** Compute the sample covariance matrix

\[
C := \text{Cov}(\Delta) = N^{-1} \sum_{t=1}^{N} \Delta(t)\Delta(t)^\top
\]

\[
= \left( N^{-1} \sum_{t=1}^{n} \Delta_i(t)\Delta_j(t) \right)_{i,j=1,\ldots,n}.
\]

3. **Diagonalization.** Decompose \( C \) as \( C = U^\top \Lambda U \), where
   - \( \Lambda \) is the \( n \times n \) diagonal matrix gathering the eigenvalues of \( C \) in a decreasing order, i.e.,
     \[
     \Lambda = \text{Diag}(\lambda_1, \ldots, \lambda_n), \quad \text{with } \lambda_1 > \cdots > \lambda_n,
     \]
   - \( U \) is the \( n \times n \) matrix assembling the corresponding normalized eigenvectors column by column, i.e.,
     \[
     U = (u_1^\top | \ldots | u_n^\top), \quad \text{with } \lambda_i u_i^\top = Cu_i^\top \text{ for all } i.
     \]
4. **Principal components.** For each \( i = 1, \ldots, n \), define the \( i \)th principal component \( f_i \) as the variance normalized linear combination of annualized variations with weights given by the elements \( u_1^i, \ldots, u_n^i \) of the \( i \)th eigenvector \( u_i^\top \):
\[ f = \text{Diag}(\sqrt{\lambda_1^{-1}}, \ldots, \sqrt{\lambda_n^{-1}}) U^\top \Delta. \]

In matrix-like form, this expression reads as

\[
\begin{pmatrix}
  f_1 \\
  \vdots \\
  f_n
\end{pmatrix}
= \left( \sum_{k=1}^n \sqrt{\lambda_1^{-1}} u_1^k \Delta_k \right)
= \left( \sum_{k=1}^n \sqrt{\lambda_2^{-1}} u_2^k \Delta_k \right)
= \cdots
= \left( \sum_{k=1}^n \sqrt{\lambda_n^{-1}} u_n^k \Delta_k \right),
\]

where \( \Delta_k \) is the \( k \)th entry of vector \( \Delta \). As a linear combination of normal variables with zero mean, vector \( f \) is also normal with mean equal to the zero vector \( \mathbf{0} = (0, \ldots, 0)^\top \). Moreover, its covariance is given by the \( n \times n \) identity matrix, that is,

\[
\text{Cov}(f) = \text{Cov}(\text{Diag}(\sqrt{\lambda_1^{-1}}, \ldots, \sqrt{\lambda_n^{-1}}) U^\top \Delta)
= \text{Diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) U \text{ Cov}(\Delta) U^\top
= \text{Diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) U C U^\top
= \text{Diag}(\lambda_1^{-1}, \ldots, \lambda_n^{-1}) U U^\top \Lambda U^\top
= I_n,
\]

where we applied the property that the transpose of an orthogonal matrix is equal to its inverse, e.g., \( U^\top = U^{-1} \). In short, \( f \sim \mathcal{N}_n(\mathbf{0}, I_n) \). Since \( \lambda_i > \lambda_{i+1} \), \( f_i \) can be interpreted as the \( i \)th most important component in explaining the cross-sectional risk \( C \).

5. **Diffusion coefficients identification.** By setting \( dt = \delta \) and \( dW(t) = \Delta \) in equation (23.3), and by noticing that

\[
\Delta = \text{Diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}) U f = \left( \begin{array}{c}
\sum_{j=1}^n \sqrt{\lambda_1} u_1^j f_j \\
\vdots \\
\sum_{j=1}^n \sqrt{\lambda_n} u_n^j f_j
\end{array} \right),
\]

we come up to the following set of equations:

\[
\begin{align*}
\frac{dy_1(t)}{y_1(t)} &= \mu_1 \, dt + \sum_{k=1}^n \sqrt{\lambda_1} u_1^k \, d\tilde{W}_k(t), \\
\frac{dy_2(t)}{y_2(t)} &= \mu_2 \, dt + \sum_{k=1}^n \sqrt{\lambda_2} u_2^k \, d\tilde{W}_k(t), \\
&\vdots \\
\frac{dy_n(t)}{y_n(t)} &= \mu_n \, dt + \sum_{k=1}^n \sqrt{\lambda_n} u_n^k \, d\tilde{W}_k(t).
\end{align*}
\]

where \( d\tilde{W}_j := f_j \). By reducing the number of factors to \( m < n \), the resulting dynamic system reads as in equation (23.2), where \( \beta_{ik} := \sqrt{\lambda_i} u_i^k \).
The entries of each eigenvector $\mathbf{u}^i$ are referred to as “factor loadings”. Indeed, formula (23.6) implies that a unit perturbation in the component $\sqrt{\lambda_i} f_i$ generates a shock $u_1^i$ on the yield $y_1$, a shock $u_2^i$ on the yield $y_2$, and so on. Usually, the entries of the first eigenvector display a common order of magnitude and share the same sign. This means that a unit shock in the first component is reflected in a similar movement of all points on the curve, that is a parallel shift. Similar interpretations can be attributed to some of the remaining factors, as will be clear at the end of our analysis.

The first applications of this decomposition in finance go back to Steeley (1990) and Litterman and Scheinkman (1991). These authors studied the yield curve dynamics of the US interest rate market. Their main conclusion was that the first three components $f_1^1$, $f_2^2$ and $f_3^3$ explain more than 90% of the historical market volatility during the eighties. Moreover, these components can be respectively interpreted as a parallel shift, a change in slope, and a convexity adjustment in the shape of the term structure. Although this method has been criticized for the strong assumption made about the normality of the distribution of centered yield returns, it has now become a market standard in the banking industry for the purpose of modelling cross-sectional risk.

23.2.2 Code

Our implementation involves two codes, one for the scenario simulation model and another for the Monte Carlo simulation. Centered yield variations are calculated for a set of times-to-maturity. Function $\text{cov}$ performs and delivers the variance–covariance matrix of the time series of these quantities. Function $\text{eig}$ returns the set of decreasingly ordered eigenvalues and the corresponding normalized eigenvectors. Once PCA has been performed, the four most significant components are singled out and sample yield curves are generated.

The two codes differ in the way these curves are obtained. The scenario simulation code uses a finite set of possible outcomes for each of the four components. The Monte Carlo code generates pseudo-random samples for these figures. Once outcomes for the selected components have been obtained, their value is plugged into formula (23.7) and a set of sample yields, one per time-to-maturity, is returned as an output.

23.3 Results and Comments

We implemented and tested the simulation algorithm using a time series of yield curves in the US Treasury bond market. Our data span the period from March 1, 2000, to February 28, 2002, and the cross-section includes eleven yields, namely those referring to yearly times-to-maturity for years one to ten, plus the six-month yield.

Recall that the $j$th factor is given by a constant multiplied by the linear combination of yield increments $\Delta_1, \ldots, \Delta_{11}$ defined in (23.5) with loadings $u_1^j, \ldots, u_{11}^j$, ...
i.e., the entries of the normalized eigenvector corresponding to the $j$th greatest eigenvalue of the yield increments covariance matrix. We may figure out the shape of a given factor $j$ by visually inspecting the graph of all corresponding factor loadings plotted against the time-to-maturity variable, i.e., $\{(T_i, u_j^T) | i = 1, \ldots, 11\}$. Figure 23.1 shows a graph of factor loadings defining the most significant four factors. We see that the first factor assigns quite uniform loading coefficients to all yield increments. This feature suggests that the most important component driving the yield curve risk is a parallel shift in the curve shape. The second factor loadings display an increasing path. Correspondingly, this factor can be interpreted as a change in the yield curve slope. Finally, the third factor can be identified with a curvature change in the yield curve shape.

Table 23.1 reports instantaneous annualized standard deviations for the 11 principal components. The overall variance is given by the sum of the diagonal entries in the covariance matrix $R$ or, equivalently, by the sum of all its eigenvalues$^2$

$$\text{Var}_{\text{Tot}} = 0.153370 + 0.018219 + \cdots + 0.000003 = 0.174838.$$  

$^2$Since the trace operator $\text{Tr}(A) := \sum_i A_{ii}$ is invariant under orthogonal transformations and commutation among arguments, we have:

$$\sum_i R_{ij} \triangleq \text{Tr}(R) = \text{Tr}(\Sigma A \Sigma^T) = \text{Tr}(\Sigma \Sigma^T A) \quad \Sigma^T = \Sigma^{-1} \quad \text{Tr}(A) \triangleq \sum_i \lambda_i.$$
### Table 23.1. Instantaneous annualized volatilities reproduced by the eleven principal components

<table>
<thead>
<tr>
<th>Factor</th>
<th>Label</th>
<th>Variance, $\lambda_j$</th>
<th>Std. dev., $\sqrt{\lambda_j}$</th>
<th>% Var., $\lambda_j / \sum_{k=1}^{11} \lambda_k$</th>
<th>% Cumul., $\sum_{k=1}^{j} \lambda_k / \sum_{k=1}^{11} \lambda_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\tilde{W}_1$</td>
<td>0.153370</td>
<td>3.916%</td>
<td>87.721%</td>
<td>87.721%</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{W}_2$</td>
<td>0.018219</td>
<td>1.350%</td>
<td>10.420%</td>
<td>98.141%</td>
</tr>
<tr>
<td>3</td>
<td>$\tilde{W}_3$</td>
<td>0.002309</td>
<td>0.481%</td>
<td>1.321%</td>
<td>99.462%</td>
</tr>
<tr>
<td>4</td>
<td>$\tilde{W}_4$</td>
<td>0.000585</td>
<td>0.242%</td>
<td>0.335%</td>
<td>99.796%</td>
</tr>
<tr>
<td>5</td>
<td>$\tilde{W}_5$</td>
<td>0.000181</td>
<td>0.135%</td>
<td>0.104%</td>
<td>99.900%</td>
</tr>
<tr>
<td>6</td>
<td>$\tilde{W}_6$</td>
<td>0.000137</td>
<td>0.117%</td>
<td>0.078%</td>
<td>99.978%</td>
</tr>
<tr>
<td>7</td>
<td>$\tilde{W}_7$</td>
<td>0.000017</td>
<td>0.041%</td>
<td>0.010%</td>
<td>99.988%</td>
</tr>
<tr>
<td>8</td>
<td>$\tilde{W}_8$</td>
<td>0.000008</td>
<td>0.028%</td>
<td>0.005%</td>
<td>99.992%</td>
</tr>
<tr>
<td>9</td>
<td>$\tilde{W}_9$</td>
<td>0.000005</td>
<td>0.023%</td>
<td>0.003%</td>
<td>99.995%</td>
</tr>
<tr>
<td>10</td>
<td>$\tilde{W}_{10}$</td>
<td>0.000004</td>
<td>0.019%</td>
<td>0.002%</td>
<td>99.997%</td>
</tr>
<tr>
<td>11</td>
<td>$\tilde{W}_{11}$</td>
<td>0.000003</td>
<td>0.018%</td>
<td>0.002%</td>
<td>100.000%</td>
</tr>
</tbody>
</table>

**Fig. 23.2.** Quota of the sample variance reported as a function of the cumulative number of ranked principal components (factors).

In particular, the first component accounts for $0.153370^2/0.174838 \approx 87.721\%$ of the total variance. Figure 23.2 displays the cumulated variance reproduced by an increasing number of principal components. It is clear from this picture that the most significant four components represent almost the entire sample variance in the market under investigation. Accordingly, we select $\tilde{W}_1$, $\tilde{W}_2$, $\tilde{W}_3$ and $\tilde{W}_4$ for the purpose of generating sample yield curves.
Our next experiment involves the yield curve as recorded on February 28, 2002, that is the last available datum in our set. We then generate eleven yields defining a sample yield curve as will be quoted in three months from the current date. This task is accomplished through standard Monte Carlo and scenario simulation. To put this program into action, let $t_0$ and $(y_1(t_0), \ldots, y_{11}(t_0))$ denote the starting date and the quoted term structure, respectively. If the simulation horizon is $t = t_0 + 3$ months, a standard Monte Carlo sample $(y_1(t), \ldots, y_{11}(t))$ is obtained as

$$y_i(t) = y_i(t_0) \exp \left[ 1 + \mu_i(t - t_0) + \sum_{k=1}^{4} \sqrt{\lambda_i} u^k_i \sqrt{t - t_0} \times N^k(0, 1) \right],$$

where $\mu_i$ is the $i$th entry of vector $\mu$ defined in (23.4) and $N^1(0, 1), \ldots, N^4(0, 1)$ are independent samples from a standard normal distribution. A sample performed using the scenario simulation model can be obtained by assuming that each $N^k$ is a state of a discrete distribution approximating the normal density. We consider a discrete set of states $w^k_1, \ldots, w^k_m$ together with a distribution function defined by

$$P(w^k_i) = 2^{-m} \frac{m!}{i!(m-i)!}, \quad i = 1, \ldots, m, k = 1, \ldots, 4.$$
Table 23.2. Scenario simulation approximation: states and probabilities for factors 1–4

<table>
<thead>
<tr>
<th>Factor, $k$ = 1</th>
<th>$w^k_{i}$:</th>
<th>$w^1_1$</th>
<th>$w^1_2$</th>
<th>$w^1_3$</th>
<th>$w^1_4$</th>
<th>$w^1_5$</th>
<th>$w^1_6$</th>
<th>$w^1_7$</th>
<th>$w^1_8$</th>
<th>$w^1_9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p w^k_{i}$:</td>
<td>0.004</td>
<td>0.031</td>
<td>0.109</td>
<td>0.219</td>
<td>0.273</td>
<td>0.219</td>
<td>0.109</td>
<td>0.031</td>
<td>0.004</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Factor, $k$ = 2</th>
<th>$w^k_{i}$:</th>
<th>$w^2_1$</th>
<th>$w^2_2$</th>
<th>$w^2_3$</th>
<th>$w^2_4$</th>
<th>$w^2_5$</th>
<th>$w^2_6$</th>
<th>$w^2_7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p w^k_{i}$:</td>
<td>0.016</td>
<td>0.094</td>
<td>0.234</td>
<td>0.313</td>
<td>0.234</td>
<td>0.094</td>
<td>0.016</td>
<td></td>
</tr>
</tbody>
</table>

| Factor, $k$ = 3 | $w^k_{i}$: | $w^3_1$ | $w^3_2$ | $w^3_3$ | $w^3_4$ | $w^3_5$ | $w^3_6$ | $w^3_7$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $p w^k_{i}$:    | 0.062 | 0.250 | 0.370 | 0.250 | 0.062 |

<table>
<thead>
<tr>
<th>Factor, $k$ = 4</th>
<th>$w^k_{i}$:</th>
<th>$w^4_1$</th>
<th>$w^4_2$</th>
<th>$w^4_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p w^k_{i}$:</td>
<td>0.250</td>
<td>0.500</td>
<td>0.250</td>
<td></td>
</tr>
</tbody>
</table>

Fig. 23.3. Joint plot of 105 sample yield curves simulated on a 3-month horizon by standard Monte Carlo.
to implement. Secondly, their limits are well known, which may not be the case for alternative and more sophisticated methods. Finally, PCA constitutes the standard market practice. Thus, no serious comparative risk analysis should avoid this setting as a benchmark.

It is worth mentioning that traditional PCA may be extended to the analysis of cross-sectional shifts at a functional level. In this respect, Guiotto and Roncoroni (2001) propose a theoretical framework where principal components are identified in the stylized deformations observed on a time-series of cross-sectional data. The study conducted by Galluccio and Roncoroni (2006) supported this approach from an empirical viewpoint. In particular, these authors showed that factors underlying cross-sectional shifts reproduce the yield curve risk more accurately than the traditional factors derived through standard PCA. Moreover, hedging a simple liability against functional risk is more effective than hedging against the PCA factors in terms of descriptive statistics of P&amp;L distributions.

Table 23.3. Monte Carlo simulation, 4 factors

<table>
<thead>
<tr>
<th>Time-to-maturity</th>
<th>Min</th>
<th>Mean</th>
<th>Max</th>
<th>St. dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 m</td>
<td>1.785</td>
<td>2.061</td>
<td>2.245</td>
<td>0.01113</td>
</tr>
<tr>
<td>1 y</td>
<td>2.140</td>
<td>2.445</td>
<td>2.652</td>
<td>0.01438</td>
</tr>
<tr>
<td>2 y</td>
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<td>3.414</td>
<td>3.600</td>
<td>0.01351</td>
</tr>
<tr>
<td>3 y</td>
<td>3.865</td>
<td>4.071</td>
<td>4.224</td>
<td>0.01211</td>
</tr>
<tr>
<td>4 y</td>
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<td>4.511</td>
<td>4.662</td>
<td>0.01135</td>
</tr>
<tr>
<td>5 y</td>
<td>4.639</td>
<td>4.827</td>
<td>4.984</td>
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</tr>
<tr>
<td>6 y</td>
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<td>5.080</td>
<td>5.234</td>
<td>0.01046</td>
</tr>
<tr>
<td>7 y</td>
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<td>5.265</td>
<td>5.426</td>
<td>0.01024</td>
</tr>
<tr>
<td>8 y</td>
<td>5.328</td>
<td>5.414</td>
<td>5.582</td>
<td>0.01002</td>
</tr>
<tr>
<td>9 y</td>
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<td>5.530</td>
<td>5.706</td>
<td>0.00989</td>
</tr>
<tr>
<td>10 y</td>
<td>5.452</td>
<td>5.634</td>
<td>5.819</td>
<td>0.00977</td>
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Table 23.4. Scenario simulation, 4 factors

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<th>Mean</th>
<th>Max</th>
<th>St. dev.</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.077</td>
<td>2.506</td>
<td>0.01220</td>
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<tr>
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